

Due Wednesday, March 17 at the beginning of your discussion section.

You must write the solutions to the problems single-sided on your own lined paper, with all sheets stapled together, and with all answers written in sequential order or you will lose points.

1. Simplify this expression: $\ln \left[\prod_{i=1}^n e^{f(i)} \right]$.

Answer:

$$\begin{aligned} \ln \left[\prod_{i=1}^n e^{f(i)} \right] &= \ln \left[e^{f(1)} \cdot e^{f(2)} \dots e^{f(n)} \right] \\ &= \ln e^{f(1)} + \ln e^{f(2)} + \dots + \ln e^{f(n)} \\ &= f(1) + f(2) + \dots + f(n) \\ &= \sum_{i=1}^n f(i) \end{aligned}$$

2. Prove $\forall n \in \mathbf{Z}^+ \sum_{i=1}^n (2i)(2i-1) = \frac{n(n+1)(4n-1)}{3}$.

Answer:

Base Case: ($n = 1$)

$$\text{(LHS)} \sum_{i=1}^1 (2i)(2i-1) = (2)(2-1) = 2$$

$$\text{(RHS)} \frac{1(1+1)(4-1)}{3} = \frac{2(3)}{3} = 2$$

$$2 = 2 \quad \checkmark$$

Inductive Hypothesis: ($n = k$)

$$\sum_{i=1}^k (2i)(2i-1) = \frac{k(k+1)(4k-1)}{3}$$

Inductive Step: ($n = k+1$)

$$\text{Show: } \sum_{i=1}^{k+1} (2i)(2i-1) = \frac{(k+1)((k+1)+1)(4(k+1)-1)}{3}$$

Proof:

$$\begin{aligned}
\sum_{i=1}^{k+1} (2i)(2i-1) &= \sum_{i=1}^k (2i)(2i-1) + \sum_{i=k+1}^{k+1} (2i)(2i-1) \\
&= \sum_{i=1}^k (2i)(2i-1) + 2(k+1)(2(k+1)-1) \\
&= \frac{k(k+1)(4k-1)}{3} + 2(k+1)(2k+1) && \text{(by the IH)} \\
&= \frac{k(k+1)(4k-1)}{3} + \frac{6(k+1)(2k+1)}{3} \\
&= \frac{(k+1)[k(4k-1) + 6(2k+1)]}{3} \\
&= \frac{(k+1)[4k^2 + 11k + 6]}{3} \\
&= \frac{(k+1)(k+2)(4k+3)}{3} \\
&= \frac{(k+1)((k+1)+1)(4(k+1)-1)}{3} \quad \checkmark
\end{aligned}$$

3. Prove $\forall n \in \mathbf{Z}^+ \sum_{i=1}^n (4i-3) = n(2n-1)$.

Answer:

Base Case: ($n = 1$)

$$\begin{aligned}
\text{(LHS)} \quad \sum_{i=1}^1 (4i-3) &= 4(1) - 3 = 1 \\
\text{(RHS)} \quad n(2n-1) &= 1(2-1) = 1 \\
1 &= 1 \quad \checkmark
\end{aligned}$$

Inductive Hypothesis: ($n = k$)

$$\sum_{i=1}^k (4i-3) = k(2k-1)$$

Inductive Step: ($n = k+1$)

$$\text{Show: } \sum_{i=1}^{k+1} (4i-3) = (k+1)(2(k+1)-1)$$

Proof:

$$\begin{aligned}
\sum_{i=1}^{k+1} (4i-3) &= \sum_{i=1}^k (4i-3) + \sum_{i=k+1}^{k+1} (4i-3) \\
&= \sum_{i=1}^k (4i-3) + (4(k+1)-3) \\
&= k(2k-1) + 4(k+1) - 3 && \text{(by the IH)} \\
&= 2k^2 - k + 4k + 4 - 3 = 2k^2 + 3k + 1 \\
&= (k+1)(2k+1) = (k+1)(2(k+1)-1) \quad \checkmark
\end{aligned}$$

4. Prove $\forall n \in \mathbf{Z}^+ \prod_{i=1}^n 2^i = 2^{\left(\frac{n}{2} + \frac{n^2}{2}\right)}$.

Answer:

Base Case: ($n = 1$)

$$(\text{LHS}) \prod_{i=1}^1 2^i = 2$$

$$(\text{RHS}) 2^{\left(\frac{1}{2} + \frac{1^2}{2}\right)} = 2^1 = 2$$

$$2 = 2 \quad \checkmark$$

Inductive Hypothesis: ($n = k$)

$$\prod_{i=1}^k 2^i = 2^{\left(\frac{k}{2} + \frac{k^2}{2}\right)}$$

Inductive Step: ($n = k + 1$)

$$\text{Show: } \prod_{i=1}^{k+1} 2^i = 2^{\left(\frac{k+1}{2} + \frac{(k+1)^2}{2}\right)}$$

Proof:

$$\begin{aligned} \prod_{i=1}^{k+1} 2^i &= \left(\prod_{i=1}^k 2^i \right) \left(\prod_{i=k+1}^{k+1} 2^i \right) = \left(\prod_{i=1}^k 2^i \right) (2^{k+1}) \\ &= \left(2^{\left(\frac{k}{2} + \frac{k^2}{2}\right)} \right) (2^{k+1}) && (\text{by the IH}) \\ &= 2^{\frac{k}{2} + \frac{k^2}{2} + k+1} = 2^{\frac{k+k^2+2k+2}{2}} \\ &= 2^{\frac{(k+1) + (k^2+2k+1)}{2}} = 2^{\left(\frac{k+1}{2} + \frac{(k+1)^2}{2}\right)} \quad \checkmark \end{aligned}$$

5. Recall the recursive definition of the *Fibonacci sequence*:

$$F_1 = 1$$

$$F_2 = 1$$

$$F_k = F_{k-1} + F_{k-2} \quad \text{for } k > 2.$$

Let's perform a change of variable on the third line to get $F_{m+2} = F_m + F_{m+1}$ for $m > 0$, and $F_{n+3} = F_{n+1} + F_{n+2}$ for $n \geq 0$.

Prove these interesting facts about this sequence:

$$(a) \quad \forall n \in \mathbf{Z}^+ \sum_{k=1}^n F_k = F_{n+2} - 1$$

Answer:

Base Case: ($n = 1$)

$$(\text{LHS}) \sum_{k=1}^1 F_k = F_1 = 1$$

$$(\text{RHS}) F_{1+2} - 1 = F_3 - 1 = 2 - 1 = 1$$

$$1 = 1 \quad \checkmark$$

Inductive Hypothesis: ($n = p$)

$$\sum_{k=1}^p F_k = F_{p+2} - 1$$

Inductive Step: ($n = p + 1$)

Show: $\sum_{k=1}^{p+1} F_k = F_{p+3} - 1$

Proof:

$$\begin{aligned} \sum_{k=1}^{p+1} F_k &= \sum_{k=1}^p F_k + \sum_{k=p+1}^{p+1} F_k = \sum_{k=1}^p F_k + F_{p+1} \\ &= F_{p+2} - 1 + F_{p+1} && \text{(by the IH)} \\ &= F_{p+1} + F_{p+2} - 1 \\ &= F_{p+3} - 1 && \text{(by the definition of the Fib. seq.)} \end{aligned}$$

(b) $\forall n \in \mathbf{Z}^+ \quad \sum_{k=1}^n F_k^2 = F_n \cdot F_{n+1}.$

Answer:

Base Case: ($n = 1$)

$$\begin{aligned} \text{(LHS)} \quad \sum_{k=1}^1 F_k^2 &= F_1^2 = 1 \\ \text{(RHS)} \quad F_1 \cdot F_{1+1} &= F_1 \cdot F_2 = 1 \cdot 1 = 1 \\ 1 &= 1 \quad \checkmark \end{aligned}$$

Inductive Hypothesis: ($n = p$)

$$\sum_{k=1}^p F_k^2 = F_p \cdot F_{p+1}$$

Inductive Step: ($n = p + 1$)

Show: $\sum_{k=1}^{p+1} F_k^2 = F_{p+1} \cdot F_{p+2}$

Proof:

$$\begin{aligned} \sum_{k=1}^{p+1} F_k^2 &= \sum_{k=1}^p F_k^2 + \sum_{k=p+1}^{p+1} F_k^2 = \sum_{k=1}^p F_k^2 + F_{p+1}^2 \\ &= F_p \cdot F_{p+1} + F_{p+1}^2 && \text{(by the IH)} \\ &= F_{p+1} (F_p + F_{p+1}) \\ &= F_{p+1} \cdot F_{p+2} \quad \checkmark && \text{(by the definition of the Fib. seq.)} \end{aligned}$$