CMSC250, Spring 2004 Homework 7 Answers

Due Wednesday, March 17 at the beginning of your discussion section.

You must write the solutions to the problems single-sided on your own lined paper, with all sheets stapled together, and with all answers written in sequential order or you will lose points.

1. Simplify this expression: $\ln\left[\prod_{i=1}^{n} e^{f(i)}\right]$.

   Answer:

$$\begin{aligned}
\ln\left[\prod_{i=1}^{n} e^{f(i)}\right] &= \ln\left[e^{f(1)} \cdot e^{f(2)} \cdots e^{f(n)}\right] \\
&= \ln e^{f(1)} + \ln e^{f(2)} + \cdots + \ln e^{f(n)} \\
&= f(1) + f(2) + \cdots + f(n) \\
&= \sum_{i=1}^{n} f(i)
\end{aligned}$$

2. Prove $\forall n \in \mathbf{Z}^+ \ \sum_{i=1}^{n}(2i)(2i-1) = \frac{n(n+1)(4n-1)}{3}$.

   Answer:

   Base Case: $(n = 1)$

   (LHS) $\sum_{i=1}^{1}(2i)(2i-1) = (2)(2-1) = 2$

   (RHS) $\frac{1(1+1)(4-1)}{3} = \frac{2(3)}{3} = 2$

   $2 = 2 \quad \surd$

   Inductive Hypothesis: $(n = k)$

   $\sum_{i=1}^{k}(2i)(2i-1) = \frac{k(k+1)(4k-1)}{3}$

   Inductive Step: $(n = k + 1)$

   Show: $\sum_{i=1}^{k+1}(2i)(2i-1) = \frac{(k+1)((k+1)+1)(4(k+1)-1)}{3}$

Proof:

$$\begin{aligned}
\sum_{i=1}^{k+1}(2i)(2i-1) &= \sum_{i=1}^{k}(2i)(2i-1) + \sum_{i=k+1}^{k+1}(2i)(2i-1) \\
&= \sum_{i=1}^{k}(2i)(2i-1) + 2(k+1)(2(k+1)-1) \\
&= \frac{k(k+1)(4k-1)}{3} + 2(k+1)(2k+1) \qquad \text{(by the IH)} \\
&= \frac{k(k+1)(4k-1)}{3} + \frac{6(k+1)(2k+1)}{3} \\
&= \frac{(k+1)\left[k(4k-1)+6(2k+1)\right]}{3} \\
&= \frac{(k+1)\left[4k^2+11k+6\right]}{3} \\
&= \frac{(k+1)(k+2)(4k+3)}{3} \\
&= \frac{(k+1)((k+1)+1)(4(k+1)-1)}{3} \quad \surd
\end{aligned}$$

3. Prove $\forall n \in \mathbf{Z}^+ \ \sum_{i=1}^{n}(4i-3) = n(2n-1)$.

Answer:

Base Case: $(n=1)$

(LHS) $\sum_{i=1}^{1}(4i-3) = 4(1)-3 = 1$

(RHS) $n(2n-1) = 1(2-1) = 1$

$1 = 1 \quad \surd$

Inductive Hypothesis: $(n=k)$

$\sum_{i=1}^{k}(4i-3) = k(2k-1)$

Inductive Step: $(n=k+1)$

Show: $\sum_{i=1}^{k+1}(4i-3) = (k+1)(2(k+1)-1)$

Proof:

$$\begin{aligned}
\sum_{i=1}^{k+1}(4i-3) &= \sum_{i=1}^{k}(4i-3) + \sum_{i=k+1}^{k+1}(4i-3) \\
&= \sum_{i=1}^{k}(4i-3) + (4(k+1)-3) \\
&= k(2k-1) + 4(k+1) - 3 \qquad \text{(by the IH)} \\
&= 2k^2 - k + 4k + 4 - 3 = 2k^2 + 3k + 1 \\
&= (k+1)(2k+1) = (k+1)(2(k+1)-1) \quad \surd
\end{aligned}$$

4. Prove $\forall n \in \mathbf{Z}^+ \ \prod_{i=1}^{n} 2^i = 2^{\left(\frac{n}{2}+\frac{n^2}{2}\right)}$.

Answer:

Base Case: $(n = 1)$

(LHS) $\prod_{i=1}^{1} 2^i = 2$

(RHS) $2^{\left(\frac{1}{2}+\frac{1}{2}\right)} = 2^1 = 2$

$2 = 2 \quad \surd$

Inductive Hypothesis: $(n = k)$

$\prod_{i=1}^{k} 2^i = 2^{\left(\frac{k}{2}+\frac{k^2}{2}\right)}$

Inductive Step: $(n = k + 1)$

Show: $\prod_{i=1}^{k+1} 2^i = 2^{\left(\frac{k+1}{2}+\frac{(k+1)^2}{2}\right)}$

Proof:

$$
\begin{aligned}
\prod_{i=1}^{k+1} 2^i &= \left(\prod_{i=1}^{k} 2^i\right)\left(\prod_{i=k+1}^{k+1} 2^i\right) = \left(\prod_{i=1}^{k} 2^i\right)\left(2^{k+1}\right) \\
&= \left(2^{\left(\frac{k}{2}+\frac{k^2}{2}\right)}\right)\left(2^{k+1}\right) && \text{(by the IH)} \\
&= 2^{\frac{k}{2}+\frac{k^2}{2}+k+1} = 2^{\frac{k+k^2+2k+2}{2}} \\
&= 2^{\frac{(k+1)+\left(k^2+2k+1\right)}{2}} = 2^{\left(\frac{k+1}{2}+\frac{(k+1)^2}{2}\right)} \quad \surd
\end{aligned}
$$

5. Recall the recursive definition of the *Fibonacci sequence*:

$$
\begin{aligned}
F_1 &= 1 \\
F_2 &= 1 \\
F_k &= F_{k-1} + F_{k-2} \quad \text{for } k > 2.
\end{aligned}
$$

Let's perform a change of variable on the third line to get $F_{m+2} = F_m + F_{m+1}$ for $m > 0$, and $F_{n+3} = F_{n+1} + F_{n+2}$ for $n \geq 0$.

Prove these interesting facts about this sequence:

(a) $\forall n \in \mathbf{Z}^+ \ \sum_{k=1}^{n} F_k = F_{n+2} - 1$

Answer:

Base Case: $(n = 1)$

(LHS) $\sum_{k=1}^{1} F_k = F_1 = 1$

(RHS) $F_{1+2} - 1 = F_3 - 1 = 2 - 1 = 1$

$1 = 1 \quad \surd$

Inductive Hypothesis: $(n = p)$

$$\sum_{k=1}^{p} F_k = F_{p+2} - 1$$

Inductive Step: $(n = p + 1)$

Show: $\displaystyle\sum_{k=1}^{p+1} F_k = F_{p+3} - 1$

Proof:

$$\begin{aligned}
\sum_{k=1}^{p+1} F_k = \sum_{k=1}^{p} F_k + \sum_{k=p+1}^{p+1} F_k &= \sum_{k=1}^{p} F_k + F_{p+1} \\
&= F_{p+2} - 1 + F_{p+1} && \text{(by the IH)} \\
&= F_{p+1} + F_{p+2} - 1 \\
&= F_{p+3} - 1 && \text{(by the definition of the Fib. seq.)}
\end{aligned}$$

(b) $\forall n \in \mathbf{Z}^+ \ \displaystyle\sum_{k=1}^{n} F_k^2 = F_n \cdot F_{n+1}$.

Answer:

Base Case: $(n = 1)$

(LHS) $\displaystyle\sum_{k=1}^{1} F_k^2 = F_1^2 = 1$

(RHS) $F_1 \cdot F_{1+1} = F_1 \cdot F_2 = 1 \cdot 1 = 1$

$1 = 1 \quad \surd$

Inductive Hypothesis: $(n = p)$

$$\sum_{k=1}^{p} F_k^2 = F_p \cdot F_{p+1}$$

Inductive Step: $(n = p + 1)$

Show: $\displaystyle\sum_{k=1}^{p+1} F_k^2 = F_{p+1} \cdot F_{p+2}$

Proof:

$$\begin{aligned}
\sum_{k=1}^{p+1} F_k^2 = \sum_{k=1}^{p} F_k^2 + \sum_{k=p+1}^{p+1} F_k^2 &= \sum_{k=1}^{p} F_k^2 + F_{p+1}^2 \\
&= F_p \cdot F_{p+1} + F_{p+1}^2 && \text{(by the IH)} \\
&= F_{p+1} (F_p + F_{p+1}) \\
&= F_{p+1} \cdot F_{p+2} \quad \surd && \text{(by the definition of the Fib. seq.)}
\end{aligned}$$