CMSC250, Spring 2004 — Homework 8 Answers

Due Wednesday, March 31 at the beginning of your discussion section.

You must write the solutions to the problems single-sided on your own lined paper, with all sheets stapled together, and with all answers written in sequential order or you will lose points.

1. Prove $\forall n \in \mathbf{Z}^+ \ \sum_{i=1}^{n} \left\lceil \frac{i}{2} \right\rceil > \frac{(n-1)(n+2)}{4}$.

   Answer:

   Base Case: $(n = 1)$

   (LHS) $\sum_{i=1}^{1} \left\lceil \frac{i}{2} \right\rceil = \lceil 1/2 \rceil = 1$

   (RHS) $\frac{(n-1)(n+2)}{4} = \frac{(1-1)(1+2)}{4} = 0$

   $1 > 0 \quad \surd$

   Inductive Hypothesis: $(n = k)$

   $\sum_{i=1}^{k} \left\lceil \frac{i}{2} \right\rceil > \frac{(k-1)(k+2)}{4}$

   Inductive Step: $(n = k+1)$

   Show: $\sum_{i=1}^{k+1} \left\lceil \frac{i}{2} \right\rceil > \frac{((k+1)-1)((k+1)+2)}{4} = \frac{k(k+3)}{4}$

   Proof:

$$\sum_{i=1}^{k+1} \left\lceil \frac{i}{2} \right\rceil = \sum_{i=1}^{k} \left\lceil \frac{i}{2} \right\rceil + \sum_{i=k+1}^{k+1} \left\lceil \frac{i}{2} \right\rceil = \sum_{i=1}^{k} \left\lceil \frac{i}{2} \right\rceil + \left\lceil \frac{k+1}{2} \right\rceil$$
$$> \frac{(k-1)(k+2)}{4} + \left\lceil \frac{k+1}{2} \right\rceil \qquad \text{(by IH)}$$

   By the definition of ceiling, we know $\left\lceil \frac{k+1}{2} \right\rceil - 1 < \frac{k+1}{2} \leq \left\lceil \frac{k+1}{2} \right\rceil$.

   Specifically, by reversing one of the inequalities, we know $\left\lceil \frac{k+1}{2} \right\rceil \geq \frac{k+1}{2}$.

   So we can substitute $\frac{k+1}{2}$ for $\left\lceil \frac{k+1}{2} \right\rceil$.

$$\sum_{i=1}^{k+1} \left\lceil \frac{i}{2} \right\rceil > \frac{(k-1)(k+2)}{4} + \frac{k+1}{2} = \frac{(k-1)(k+2)}{4} + \frac{2+2}{4}$$
$$> \frac{k^2+k-2+2k+2}{4} = \frac{k^2+3k}{4} = \frac{k(k+3)}{4} \quad \surd$$

2. Prove $\forall n \in \mathbf{Z}^+ \ 8 \mid \left(2^{3n+1}\right)$.

   Answer: Let n be an arbitrary positive integer.
   $2^{3n+1} = 2^3 \cdot 2^{3n-2} = 8 \cdot 2^{3n-2}$.
   Since $3n-2$ is nonnegative (since $n \geq 1$),

2^{3n-2} is an integer since the integers are closed under multiplication.
Therefore, by the definition of divides,
$8 \mid \left(2^{3p+1}\right)$.

3. Prove $\forall n \in \mathbf{Z}^+ \; 8 \mid \left(3^{2n} + 7\right)$.

   Answer:

   Base Case: $(n = 1)$
   $3^{2(1)} + 7 = 9 + 7 = 16$, and $8 \mid 16$. $\surd$

   Inductive Hypothesis: $(n = k)$
   $8 \mid \left(3^{2k} + 7\right)$

   Inductive Step: $(n = k + 1)$

   Show: $8 \mid \left(3^{2(k+1)} + 7\right)$

   Proof:
   $3^{2(k+1)} + 7 = 3^{2k+2} + 7 = 9 \cdot 3^{2k} + 7$
   Now add a convenient form of zero to the right side:
   $3^{2(k+1)} + 7 = 9 \cdot 3^{2k} + 7 + [8 \cdot 7 - 8 \cdot 7]$
   $3^{2(k+1)} + 7 = 9 \cdot 3^{2k} + 9 \cdot 7 - 8 \cdot 7$
   $3^{2(k+1)} + 7 = 9\left(3^{2k} + 7\right) - 8 \cdot 7$
   By the IH, we know $8 \mid (3^{2k} + 7)$.
   By the definition of divides, $\exists p \in \mathbf{Z}$ such that $8p = 3^{2k} + 7$.
   $3^{2(k+1)} + 7 = 9\,(8p) - 8 \cdot 7 = 8(9p - 7)$.
   Since $9p - 7 \in \mathbf{Z}$ by closure of the integers under addition and multiplication,
   $8 \mid \left(3^{2(k+1)} + 7\right)$.

   Alternate Proof: Here's another way of doing the "proof" part.
   $3^{2(k+1)} + 7 = 3^{2k+2} + 7 = 9 \cdot 3^{2k} + 7$
   By the IH, we know $8 \mid (3^{2k} + 7)$.
   By the definition of divides, $\exists p \in \mathbf{Z}$ such that $8p = 3^{2k} + 7$.
   So $3^{2k} = 8p - 7$ by algebra.
   $3^{2(k+1)} + 7 = 9 \cdot 3^{2k} + 7 = 9\,(8p - 7) + 7 = 72p - 63 + 7 = 72p - 56 = 8(9p - 7)$.
   Since $9p - 7 \in \mathbf{Z}$ by closure of the integers under addition and multiplication,
   $8 \mid \left(3^{2(k+1)} + 7\right)$.

4. Prove $\forall n \in \mathbf{Z}^+ \; 8 \mid \left(2^{3n+1} + 3^{2n} - 1\right)$.

   Answer: We give three different solutions to this problem. The first one doesn't even use induction!

   Solution 1:
   Let n be an arbitrary positive integer.
   Realize that $2^{3n+1} + 3^{2n} - 1 = 2^{3n+1} + 3^{2n} - 1 + (8 - 8) = 2^{3n+1} + 3^{2n} + 7 - 8$.
   Since $n \in \mathbf{Z}^+$, we can use question 2 and state $8 \mid (2^{3n+1})$.
   Since $n \in \mathbf{Z}^+$, we can use question 3 and state $8 \mid (3^{2n} + 7)$.
   Therefore, $\exists p, q \in \mathbf{Z}$ such that $8p = 2^{3n+1}$ and $8q = 3^{2n} + 7$ by definition of divides.
   Then $2^{3n+1} + 3^{2n} + 7 - 8 = 8p + 8q - 8 = 8(p + q - 1)$ by substitution.
   Since $p + q - 1 \in \mathbf{Z}$ by closure of $\mathbf{Z}$ under addition,
   $8 \mid \left(2^{3n+1} + 3^{2n} - 1\right)$ by definition of divides.

   Solution 2:

Base Case: $(n = 1)$

$2^4 + 3^2 - 1 = 24$, and $8 \mid 24$. $\surd$

Inductive Hypothesis: $(n = k)$

$8 \mid \left(2^{3k+1} + 3^{2k} - 1\right)$

Inductive Step: $(n = k + 1)$

Show: $8 \mid \left(2^{3(k+1)+1} + 3^{2(k+1)} - 1\right)$

Proof:

$$\begin{aligned}
2^{3(k+1)+1} + 3^{2(k+1)} - 1 &= 2^{3k+4} + 3^{2k+2} - 1 \\
&= 2^{3k+1+3} + 3^{2k+2} - 1 \\
&= 2^3 \cdot 2^{3k+1} + 3^2 \cdot 3^{2k} - 1 \\
&= 8 \cdot 2^{3k+1} + 9 \cdot 3^{2k} - 1 \\
&= (9 - 1) \cdot 2^{3k+1} + 9 \cdot 3^{2k} - 1 + (9 - 9) \\
&= 9 \cdot 2^{3k+1} + 9 \cdot 3^{2k} - 9 - 2^{3k+1} + 8 \\
&= 9\left(2^{3k+1} + 3^{2k} - 1\right) - 2^{3k+1} + 8
\end{aligned}$$

Since by the IH, $8 \mid \left(2^{3k+1} + 3^{2k} - 1\right)$, $\exists p \in \mathbf{Z}$ such that $2^{3k+1} + 3^{2k} - 1 = 8p$, by definition of divides,

$$= 9(8p) - 2^{3k+1} + 8 \qquad \text{by substitution}$$

Since $k \in \mathbf{Z}^+$, by problem 2, $8 \mid 2^{3k+1}$, $\exists \ell \in \mathbf{Z}$ such that $2^{3k+1} = 8\ell$, by definition of divides,

$$\begin{aligned}
&= 9(8p) - 8\ell + 8 \qquad \text{by substitution} \\
&= 8(9p - \ell + 1)
\end{aligned}$$

Since $9p - \ell + 1 \in \mathbf{Z}$ by closure of $\mathbf{Z}$ under multiplication, subtraction, and addition, $8 \mid \left(2^{3(k+1)+1} + 3^{2(k+1)} - 1\right)$ by definition of divides.

Solution 3: Do the same induction process up until the "Proof" part...

Proof:

$$\begin{aligned}
2^{3(k+1)+1} + 3^{2(k+1)} - 1 &= 2^{3k+4} + 3^{2k+2} - 1 \\
&= 2^{3k+1+3} + 3^{2k+2} - 1 \\
&= 2^3 \cdot 2^{3k+1} + 3^2 \cdot 3^{2k} - 1 \\
&= 8 \cdot 2^{3k+1} + 9 \cdot 3^{2k} - 1 \\
&= 8 \cdot 2^{3k+1} + (8 + 1) \cdot 3^{2k} - 1 + (8 - 8) \\
&= 8 \cdot 2^{3k+1} + 8 \cdot 3^{2k} - 8 + 3^{2k} + 7 \\
&= 8\left(2^{3k+1} + 3^{2k} - 1\right) - 3^{2k} + 7
\end{aligned}$$

Since by the IH, $8 \mid \left(2^{3k+1} + 3^{2k} - 1\right)$, $\exists p \in \mathbf{Z}$ such that $2^{3k+1} + 3^{2k} - 1 = 8p$, by definition of divides,

$$= 8(8p) + 3^{2k} + 7 \qquad \text{by substitution}$$

Since $k \in \mathbf{Z}^+$, by problem 3, $8 \mid 3^{2k} + 7$, $\exists \ell \in \mathbf{Z}$ such that $3^{2k} + 7 = 8\ell$, by definition of divides,

$$\begin{aligned} &= 8(8p) + 8\ell && \text{by substitution} \\ &= 8(8p + \ell) \end{aligned}$$

Since $8p + \ell \in \mathbf{Z}$ by closure of $\mathbf{Z}$ under multiplication and addition,
$8 \mid \left(2^{3(k+1)+1} + 3^{2(k+1)} - 1\right)$ by definition of divides.

5. Let $a_0 = 2$, $a_1 = 7$, and $\forall n \in \mathbf{Z}^{>1}$ $a_n = 3a_{n-1} - 2a_{n-2}$. Prove $\forall n \in \mathbf{Z}^{\text{nonneg}}$ $a_n \equiv_5 2$.

   Answer:

   Base Case: $(n = 0, 1)$
   $a_0 = 2 \equiv_5 2 \quad \checkmark \qquad a_1 = 7 \equiv_5 2 \quad \checkmark$

   Inductive Hypothesis: $\forall i \in \mathbf{Z}$ $0 \le i < k$
   $a_i \equiv_5 2$

   Inductive Step: $(n = k)$

   Show: $a_k \equiv_5 2$

   Proof: Assume $k > 1$, since the base cases cover the other possibilities.
   Since $k > 1$, $a_k = 3a_{k-1} - 2a_{k-2}$.
   Since $0 \le k-1 < k$ and $0 \le k-2 < k$, we can invoke the inductive hypothesis on a_{k-1} and a_{k-2}: $a_{k-1} \equiv_5 2$ and $a_{k-2} \equiv_5 2$.
   Then $\exists p, q \in \mathbf{Z}$ such that $a_{k-1} = 5p + 2$ and $a_{k-2} = 5q + 2$ by definitions of divides and equivalence in a mod.
   $a_k = 3a_{k-1} - 2a_{k-2} = 3(5p + 2) - 2(5q + 2)$ by substitution.
   $a_k = 15p + 6 - 10q - 4 = 15p + 10q + 2$ by algebra.
   Then $a_k - 2 = 5(3p + 2q)$ by algebra.
   Since $3p + 2q \in \mathbf{Z}$ by closure of the integers under multiplication and addition,
   $5 \mid a_k - 2$ by definition of divides.
   $a_k \equiv_5 2$ by definition of equivalence in a mod.

6. Let $c_0 = 0$, $c_1 = 1$, and $\forall n \in \mathbf{Z}^{>1}$ $c_n = c_{\lfloor \frac{n}{2} \rfloor} + c_{\lfloor \frac{n}{3} \rfloor}$.

   Prove $\forall n \in \mathbf{Z}^{\ge 0}$ $c_n \le n$.

   Answer:

   Base Case: $(n = 0, 1)$
   $c_0 = 0 \le 0 \quad \checkmark \qquad c_1 = 1 \le 1 \quad \checkmark$

   Inductive Hypothesis: $\forall i \in \mathbf{Z}$ $0 \le i < k$
   $c_i \le i$

   Inductive Step: $(n = k)$

   Show: $c_k \le k$

   Proof: Assume $k > 1$, since the base cases cover the other possibilities.
   Since $k > 1$, $c_k = c_{\lfloor \frac{k}{2} \rfloor} + c_{\lfloor \frac{k}{3} \rfloor}$.
   Since $0 \le \left\lfloor \frac{k}{2} \right\rfloor < k$ and $0 \le \left\lfloor \frac{k}{3} \right\rfloor < k$, we can invoke the inductive hypothesis on $c_{\lfloor \frac{k}{2} \rfloor}$ and $c_{\lfloor \frac{k}{3} \rfloor}$: $c_{\lfloor \frac{k}{2} \rfloor} \le \lfloor k/2 \rfloor$ and $c_{\lfloor \frac{k}{3} \rfloor} \le \lfloor k/3 \rfloor$.
   Therefore, $c_k = c_{\lfloor \frac{k}{2} \rfloor} + c_{\lfloor \frac{k}{3} \rfloor} \le \lfloor k/2 \rfloor + \lfloor k/3 \rfloor$ by substitution.

Now we want to show that $\lfloor k/2 \rfloor + \lfloor k/3 \rfloor \leq k$, so let's assume the opposite and derive a contradiction.

Assume: $\lfloor k/2 \rfloor + \lfloor k/3 \rfloor > k$
By the definition of floor, $k/2 \geq \lfloor k/2 \rfloor$ and $k/3 \geq \lfloor k/3 \rfloor$.
Then $k/2 + k/3 > k$ by substitution.
Since $k/2 + k/3 = (5/6)k$ by algebra, $(5/6)k > k$ by substitution.
Solving for k, we get $k < 0$, which is a contradiction, since $k \in \mathbf{Z}^{\geq 0}$.

So now we know $\lfloor k/2 \rfloor + \lfloor k/3 \rfloor \leq k$, and from before, we know that $c_k \leq \lfloor k/2 \rfloor + \lfloor k/3 \rfloor$.
By the transitivity of $\leq$, we conclude that $c_k \leq k$.

7. Prove $\forall n \in \mathbf{Z}^{\text{nonneg}} \sum_{i=0}^{n} \left((2n - i)\, 2^i\right) = (n+1)\left(2^{n+1} - 2\right)$.

Answer:

Base Case: $(n = 0)$

(LHS) $\sum_{i=0}^{0} \left((2(0) - i)\, 2^i\right) = \left((2(0) - 0)\, 2^0\right) = 0$
(RHS) $(0+1)(2^{0+1} - 2) = 2 - 2 = 0$
$0 = 0$

Inductive Hypothesis: $(n = k)$

$$\sum_{i=0}^{k} \left((2k - i)\, 2^i\right) = (k+1)\left(2^{k+1} - 2\right)$$

Inductive Step: $(n = k+1)$

Show: $\sum_{i=0}^{k+1} \left((2(k+1) - i)\, 2^i\right) = ((k+1)+1)\left(2^{(k+1)+1} - 2\right)$

Proof:

$$
\begin{aligned}
\sum_{i=0}^{k+1} \left((2(k+1)-i)\,2^i\right) &= \sum_{i=0}^{k} \left((2(k+1)-i)\,2^i\right) + \sum_{i=k+1}^{k+1} \left((2(k+1)-i)\,2^i\right) \\
&= \sum_{i=0}^{k} \left((2k+2-i)\,2^i\right) + (2(k+1)-(k+1))2^{k+1} \\
&= \sum_{i=0}^{k} \left((2k-i)\,2^i + 2(2^i)\right) + (k+1)2^{k+1} \\
&= \sum_{i=0}^{k} \left((2k-i)\,2^i\right) + 2\sum_{i=0}^{k} \left(2^i\right) + (k+1)2^{k+1} \\
&= (k+1)\left(2^{k+1}-2\right) + 2\sum_{i=0}^{k} \left(2^i\right) + (k+1)2^{k+1} && \text{(by IH)} \\
&= (k+1)\left(2^{k+1}-2\right) + 2\left[2^{k+1}-1\right] + (k+1)2^{k+1} \\
&\text{(by sum of a geometric sequence)} \\
&= (k+1)\left(2^{k+1}-2+2^{k+1}\right) + \left[2^{k+2}-2\right] \\
&= (k+1)\left(2^{k+2}-2\right) + \left[2^{k+2}-2\right] \\
&= (k+1+1)\left(2^{k+2}-2\right) = ((k+1)+1)\left(2^{(k+1)+1}-2\right)
\end{aligned}
$$