CMSC250, Spring 2004 Homework 9 Answers

Due Wednesday, April 7 at the beginning of your discussion section.

You must write the solutions to the problems single-sided on your own lined paper, with all sheets stapled together, and with all answers written in sequential order or you will lose points.
Note: $\mathcal{P}(A)$ denotes the power set of A.

1. For each of the following, give a proof of the statement if it is true, or a counterexample if the statement is false. Remember, counterexamples must include specific values and enough work shown to demonstrate that they are actual counterexamples.

   (a) For all sets A, B, and C, if $A \cup C = B \cup C$, then $A = B$.

   Answer: FALSE. Counterexample:
   Let $A = \{1\}$, $B = \{1, 2\}$, and $C = \{1, 2\}$. Then $A \cup C = B \cup C = \{1, 2\}$, but $A \neq B$.

   (b) For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \times B \subseteq B \times C$.

   Answer: TRUE. Proof:
   Let A, B, and C be arbitrary sets.
   Assume $A \subseteq B \wedge B \subseteq C$.
   Let (x, y) be arbitrary in U.
   Assume $(x, y) \in A \times B$.
   $x \in A \wedge y \in B$ by definition of Cartesian product.
   Since $x \in A$ be conjunctive simplification and $A \subseteq B$,
   $x \in B$ by definition of subset.
   Since $y \in B$ be conjunctive simplification and $B \subseteq C$,
   $y \in C$ by definition of subset.
   Since $x \in B \wedge y \in C$ by conjunctive addition,
   $(x, y) \in B \times C$ by definition of Cartesian product.
   $(x, y) \in A \times B \to (x, y) \in B \times C$ by CCW.
   $\forall (x, y) \in U \ (x, y) \in A \times B \to (x, y) \in B \times C$ by generalizing from the GP.
   $A \times B \subseteq B \times C$ by definition of subset.
   $A \subseteq B \wedge B \subseteq C \to A \times B \subseteq B \times C$ by CCW.
   For all sets A, B, and C, $A \subseteq B \wedge B \subseteq C \to A \times B \subseteq B \times C$ by generalizing from the GP.

   (c) For all sets A, B, and C, $(A - B) - (B - C) = A - B$.

   Answer: TRUE. Proof: Let A, B, and C be arbitrary sets.

$$
\begin{aligned}
(A - B) - (B - C) &= (A \cap B') - (B \cap C') && \text{def of set difference} \\
&= (A \cap B') \cap (B \cap C')' && \text{def of set difference} \\
&= (A \cap B') \cap (B' \cup C) && \text{DeMorgan, double complementation} \\
&= A \cap (B' \cap (B' \cup C)) && \text{associativity} \\
&= A \cap B' && \text{absorption} \\
&= A - B && \text{def of set difference}
\end{aligned}
$$

Then for all sets A, B, and C, $(A - B) - (B - C) = A - B$ by generalizing from the GP.

(d) For all sets A and B, if $A \cap B = \emptyset$ then $A \times B = \emptyset$.

Answer: FALSE. Counterexample:
Let $A = \{1\}$ and $B = \{2\}$.
Then $A \cap B = \emptyset$, but $A \times B = \{(1, 2)\} \neq \emptyset$.

(e) For all sets A, B, and C, if $B \cap C \subseteq A$, then $(C - A) \cap (B - A) = \emptyset$.

Answer: TRUE. Proof:
Let A, B, and C be arbitrary sets.
Assume $B \cap C \subseteq A$.
We want to show that $(C - A) \cap (B - A) = \emptyset$. We know the definition of the empty set is that it has no elements. So let's assume there is some element in $(C - A) \cap (B - A)$ and derive a contradiction.
Assume $\exists x \in U$ such that $x \in (C - A) \cap (B - A)$.
$p \in (C - A) \cap (B - A)$ by existential instantiation.
$p \in (C - A) \wedge p \in (B - A)$ by definition of intersection.
$p \in (C \cap A') \wedge p \in (B \cap A')$ by definition of set difference.
$p \in C \wedge p \in A' \wedge p \in B \wedge p \in A'$ by definition of intersection.
$p \in A'$ by conjunctive simplification.
$p \in B \wedge p \in C$ by conjunctive simplification and commutativity.
$p \in (B \cap C)$ by definition of intersection.
Since $p \in (B \cap C)$ and $B \cap C \subseteq A$,
$p \in A$ by definition of subset.
$p \in A \wedge p \in A'$ by conjunctive addition. (Contradiction)
$\sim \left[\exists x \in U \text{ such that } x \in (C - A) \cap (B - A)\right]$ by CCW with contradiction.
$\forall x \in U\ x \notin (C - A) \cap (B - A)$ by DeMorgan's law.
$(C - A) \cap (B - A) = \emptyset$ by definition of the empty set (since it has no members).
For all sets A, B, and C, $[B \cap C \subseteq A] \to [(C - A) \cap (B - A) = \emptyset]$ by CCW and gen from GP.

(f) For all sets A and B, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Answer: TRUE. Proof:
Let A and B be arbitrary sets.
Let X be an arbitrary element of U such that $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$.
$X \in \mathcal{P}(A) \vee X \in \mathcal{P}(B)$ by definition of union.

Case 1: Assume $X \in \mathcal{P}(A)$.
$X \subseteq A$ by definition of membership in a power set.
Let y be an arbitrary element of U.
Assume $y \in X$.
Since $y \in X$ and $X \subseteq A$, $y \in A$ by the definition of subset.
$y \in A \vee y \in B$ by disjunctive addition.
$y \in A \cup B$ by definition of union.
$\forall y \in U\ (y \in X) \to (y \in A \cup B)$ by CCW and gen from GP.
$X \subseteq A \cup B$ by definition of subset.

Case 2: Assume $X \in \mathcal{P}(B)$.
$X \subseteq B$ by definition of membership in a power set.

Let y be an arbitrary element of U.
Assume $y \in X$.
Since $y \in X$ and $X \subseteq B$, $y \in B$ by the definition of subset.
$y \in A \vee y \in B$ by disjunctive addition.
$y \in A \cup B$ by definition of union.
$\forall y \in U \ (y \in X) \rightarrow (y \in A \cup B)$ by CCW and gen from GP.
$X \subseteq A \cup B$ by definition of subset.

Since one of these two cases always applies, and both lead to the same result, we can conclude
$X \subseteq A \cup B$ by the dilemma rule.
$X \in \mathcal{P}(A \cup B)$ by definition of membership in a power set.
$\forall X \in U \ \ [X \in \mathcal{P}(A) \cup \mathcal{P}(B)] \rightarrow [X \in \mathcal{P}(A \cup B)]$ by CCW and gen from GP.
$\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$ by definition of subset.
For all sets A and B, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$ by gen from GP.

2. Let $A_1, A_2, \ldots$ be sets. Prove the generalized DeMorgan's law:

$$\forall n \in \mathbf{Z}^+ \ (A_1 \cup A_2 \cup \cdots \cup A_n)' = A_1' \cap A_2' \cap \cdots \cap A_n'$$

Hint: Use induction, and the facts that

- $A_1 \cup A_2 \cup \cdots \cup A_n = A_1 \cup A_2 \cup \cdots \cup A_{n-1} \cup A_n$.
- $A_1 \cap A_2 \cap \cdots \cap A_n = A_1 \cap A_2 \cap \cdots \cap A_{n-1} \cap A_n$.

Answer:

Base Case: $(n = 1)$
$A_1' = B_1' \qquad \surd$

Inductive Hypothesis: $(n = k)$
Assume $(A_1 \cup A_2 \cup \cdots \cup A_k)' = A_1' \cap A_2' \cap \cdots \cap A_k'$.

Inductive Step: $(n = k + 1)$
Show: $(A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1})' = A_1' \cap A_2' \cap \cdots \cap A_k' \cap A_{k+1}'$.
Proof: Let $S = A_1 \cap A_2 \cap \cdots \cap A_k$.
Then by the IH, $S' = (A_1 \cap A_2 \cap \cdots \cap A_k)' = A_1' \cap A_2' \cap \cdots \cap A_k'$.

$$\begin{aligned}
(A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1})' &= (S \cup A_{k+1})' && \text{substitution} \\
&= S' \cap A_{k+1}' && \text{DeMorgan's law} \\
&= A_1' \cap A_2' \cap \cdots \cap A_k' \cap A_{k+1}' && \text{substitution}
\end{aligned}$$

And this is what we wanted to show.

3. Let $B_1, B_2, \ldots$ be sets.
Let $B_1 = \{0, 1\}$, $B_2 = \{1, 2\}$, and $\forall i \in \mathbf{Z}^{>2} \ B_i = (B_{i-1} \cup \{i\}) - B_{i-2}$.
Prove $\forall n \in \mathbf{Z}^+ \ B_n = \{n - 1, n\}$.

Answer:

Base Case:
$n = 1:$ $\quad B_1 = \{0, 1\}$. Since $n = 1$, $n - 1 = 0$. By substitution, $B_1 = \{n - 1, n\}$. $\quad \surd$
$n = 2:$ $\quad B_2 = \{1, 2\}$. Since $n = 2$, $n - 1 = 1$. By substitution, $B_2 = \{n - 1, n\}$. $\quad \surd$

Inductive Hypothesis: $(n = i) \quad \forall \in \mathbf{Z}\ 1 \leq i < k$

Assume $B_i = \{i-1, i\}$.

Inductive Step: $(n = k)$

Show: $B_k = \{k-1, k\}$

Proof: (Assume $k > 2$, since the other cases are covered in the base cases.)

$B_k = (B_{k-1} \cup \{k\}) - B_{k-2}$ by recursive definition.

Since $1 \leq k-1 \leq k$ and $1 \leq k-2$,

$B_{k-1} = \{(k-1)-1, k-1\}$ and $B_{k-2} = \{(k-2)-1, k-2\}$ by IH.

$B_{k-1} = \{k-2, k-1\}$ and $B_{k-2} = \{k-3, k-2\}$ by algebra.

$B_k = (\{k-2, k-1\} \cup \{k\}) - \{k-3, k-2\}$ by substitution.

$= \{k-2, k-1, k\} - \{k-3, k-2\}$ by set operations.

$= \{k-1, k\}$ by set operations.