Relations

- binary relations   xRy
- on sets $x \in X$ $y \in Y$ $R \subseteq X \times Y$
- Example:
  "less than" relation from A={0,1,2} to B={1,2,3}

<u>use traditional notation</u>

0 < 1, 0 < 2, 0 < 3, 1 < 2, 1 < 3, 2 < 3

$1 \not< 1$, $2 \not< 1$, $2 \not< 2$

<u>or use set notation</u>

A×B={(0,1),(0,2),(0,3),(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)}

R={(0,1),(0,2),(0,3), (1,2),(1,3), (2,3)}

<u>or use Arrow Diagrams</u>

Formal Definition

- (binary) relation from A to B

where $x \in A$, $y \in B$, $(x,y) \in A \times B$ and $R \subseteq A \times B$

$xRy \leftrightarrow (x,y) \in R$

- finite example: A={1,2} B={1,2,3}
- infinite example: A = Z and B = Z

$aRb \leftrightarrow a\text{-}b \in Z^{even}$

Properties of Relations

- Reflexive

R is Reflexive $\leftrightarrow \forall x \in A,\ xRx$

- Symmetric

R is Symmetric $\leftrightarrow \forall x, y \in A,\ xRy \rightarrow yRx$

- Transitive

R is Transitive $\leftrightarrow \forall x, y, z \in A,\ xRy \wedge yRz \rightarrow xRz$

Proving Properties on Infinite Sets

- $m,n \in Z$, $m \equiv_3 n$

- Reflexive
- Symmetric
- Transitive

The Inverse of the Relations and The Complement of the Relation

- $R: A \rightarrow B$ $R=\{(x,y) \in A \times B \mid xRy\}$
- $R^{-1}: B \rightarrow A$ $R^{-1} =\{(y,x) \in B \times A \mid (x,y) \in R\}$
- $R': A \rightarrow B$ $R' =\{(x,y) \in A \times B \mid (x,y) \notin R\}$

$$\forall x \in X \forall y \in Y, (y,x) \in R^{-1} \leftrightarrow (x,y) \in R$$

$$\forall x \in X \forall y \in Y, (x,y) \in R' \leftrightarrow (x,y) \notin R$$

- D : {1,2} → {2,3,4}

D={(1,2),(2,3),(2,4)} D' = ? D^{-1}=?

- $S=\{(x,y) \in R \times R \mid y = 2*|x|\}$ S^{-1}=?

Closures over the Properties

- Reflexive Closure
- Symmetric Closure
- Transitive Closure

✓ R^{xc} has property x

✓ $R \subseteq R^{xc}$

✓ R^{xc} is the minimal addition to R

(if S is any other transitive relation that contains R, $R^{xc} \subseteq S$)

Matrix Representation of a Relation

- $M_R = [m_{ij}]$ m_{ij}={1 iff (i,j) ∈R and 0 iff (i,j)∉R
- example:

R : {1,2,3} →{1,2} R = {(2,1),(3,1),(3,2)}

$$M_R = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array}$$

Powers of Relation

- For relation M,
  - M^1 = the list of paths available in 1 step
  - M^2 = the list of paths available in 2 steps
  - M^3 = the list of paths available in 3 steps
  - …
  - M^* = the list of paths available in any number of steps
- through composition
- through matrix multiplication

Union, Intersection, Difference and Composition

- R: A→B and S: A→B

$$R \bigcup S = \{(x,y) \in A \times B \mid (x,y) \in R \vee (x,y) \in S\}$$
$$R \bigcap S = \{(x,y) \in A \times B \mid (x,y) \in R \wedge (x,y) \in S\}$$
$$R - S = \{(x,y) \in A \times B \mid (x,y) \in R \wedge (x,y) \notin S\}$$
$$S - R = \{(x,y) \in A \times B \mid (x,y) \in S \wedge (x,y) \notin R\}$$

- R: A→B and S: B→C

$$S \circ R = \{(a,c) \in A \times C \mid \exists b \in B, (a,b) \in R \wedge (b,c) \in S\}$$

Equivalence Relations

- Partition the elements:

any elements “related” are in the same partition

- Equivalence Relations are
  - Reflexive
  - Symmetric
  - Transitive
- Partitions are called Equivalence Classes
  - [a] = equivalence class containing a
  - [a] = {x ∈A | xRa}

Examples

- R: X→X X={a,b,c,d,e,f}
- {(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),
  (a,e),(a,d),(d,a)(d,e),(e,a),(e,d),(b,f),(f,b)}
- Lemma 10.3.3 If A is a set and R is an equivalence relation on A and x and y are elements of A, then either

  [x] ∩ [y] = ∅ or [x] = [y]

Other Properties

Reflexive	$\forall a \in A, aRa$
Irreflexive	$\forall a \in A, a \not R a$
Symmetric	$\forall a,b \in A, aRb \to bRa$
Antisymmetric	$\forall a,b \in A, aRb \wedge bRa \to a = b$
Asymmetric	$\forall a,b \in A, aRb \to b \not R a$
Non-symmetric	$\forall a,b \in A, a \neq b \to (aRb \leftrightarrow b \not R a)$
Transitive	$\forall a,b,c \in A, aRb \wedge bRc \to aRc$

Partial Order Relation

- R is a Partial Order Relation if and only if
  R is Reflexive, Antisymmetric and Transitive
- Partial Order Set (POSET)
  (S,R) = R is a partial order relation on set S

- Examples
  – (Z,❍)
  – (Z^+,|) {note: | symbolizes divides}

Total Ordering

- When all pairs from the set are "comparable" it is called a Total Ordering

- a and b are comparable
  if and only if a R b or b R a

- a and b are non-comparable
  if and only if a $\not R$ b and b $\not R$ a

Hasse Diagram

1. take the digraph
   (since it represents the same relation)
2. arrange verticies so all arrows go upward
   (since it is antisymmetric we know this is possible)
3. remove the reflexive loops
   (since we know it is reflexive these are not necessary)
4. remove the transitive arrows
   (since we know it is transitive, these are not necessary)
5. make the remaining edges non-directed
   (since we know they are all going upward, the direction is not necessary)

Hasse Diagram Example

- The POSET ({1,2,3,9,18},|)
- The POSET ({1,2,3,4},❍)
- The POSET (P{a,b,c},$\subseteq$)

- Draw complete digraph diagrams of these relations
- Derive Hasse Diagram from those

Terminology

- Maximal : a $\in$A is maximal
  $\leftrightarrow \forall b \in A$ (bRa $\vee$ a and b are not comparable)
- Minimal : a $\in$A is minimal
  $\leftrightarrow \forall b \in A$ (aRb $\vee$ a and b are not comparable)
- Greatest : a $\in$A is greatest $\leftrightarrow$ $\forall b \in A$ (bRa)
- Least : a $\in$A is least $\leftrightarrow$ $\forall b \in A$ (aRb)

Topological Sorting

1. select any minimal
   a) put it into the list
   b) remove it from the Hasse diagram)
2. Repeat until all members are in the list

Example: ({2,3,4,6,18,24},|)