

Predicate Calculus

- Subject / Predicate

John / went to the store.

The sky / is blue.

- Propositional Logic - uses statements
- Predicate Calculus - uses predicates
 - predicates must be applied to a subject in order to be true or false
- $P(x)$
 - means this predicate represented by P
 - applied to the object represented by x

Quantification

- $\exists x$ There exists an x
- $\forall x$ For all x 's

Usually specified from a domain

- $\exists x \in \mathbb{Z}$ There exists an x in the integers
- $\forall x \in \mathbb{R}$ For all x 's in the reals
- Domain - set where these subjects come from

Translation

- **A student of mine is wearing a blue shirt.**

- Domain: people who are my students S

- Quantification: There is at least one

- Predicate: wearing a blue shirt

- $\exists x \in S$ such that $B(x)$

- where $B(x)$ represents "wearing a blue shirt"

- **My students are in class.**

- Domain: people who are my students S

- Quantification: All of them

- Predicate: are in class

- $\forall x \in S$ such that $C(x)$

- where $C(x)$ represents "being in class"

Negation of Quantified Statements

$\sim (\exists \mathbf{x} \in \mathbf{people} \text{ such that } \mathbf{H(x)})$

$\equiv \forall \mathbf{x} \in \mathbf{people} \text{ such that } \sim \mathbf{H(x)}$

$\sim(\text{There is a person who is here.}) \equiv$

For all people, each person is not here.

same in meaning as "There is no person here."

$\sim (\forall \mathbf{x} \in \mathbf{people} \text{ such that } \mathbf{H(x)})$

$\equiv \exists \mathbf{x} \in \mathbf{people} \text{ such that } \sim \mathbf{H(x)}$

$\sim(\text{For all people, each person is here.}) \equiv$

There is at least one person who is not here.

Multiple Predicate Translation

- **A student of mine is wearing a blue shirt.**

- Domain: all people P
- Quantification: There is at least one
- Predicates: "wearing a blue shirt" and "is my student"

$\exists x \in P$ such that $B(x) \wedge S(x)$

$B(x)$ represents "wearing a blue shirt"

$S(x)$ represents "being my student"

- **My students are in class.**

- Domain: all people P
- Quantification: All of them
- Predicates: "are in class" and "is my student"

$\forall x \in P$ such that $S(x) \rightarrow C(x)$

$C(x)$ represents "being in class"

$S(x)$ represents "being my student"

Multiple Quantification

- $\exists p \in P \exists c \in C, S(c,p)$
- $\exists c \in C \exists p \in P, S(c,p)$
- $\forall p \in P \forall c \in C, S(c,p)$
- $\forall c \in C \forall p \in P, S(c,p)$

where $C = \{\text{all chairs}\}$ and $P = \{\text{all people}\}$
and $S(c,p)$ represents “p sitting in c”

Mixed Multiple Quantification

- $\forall c \in C \exists p \in P, S(c,p)$
- $\forall p \in P \exists c \in C, S(c,p)$
- $\exists p \in P \forall c \in C, S(c,p)$
- $\exists c \in C \forall p \in P, S(c,p)$

where $C = \{\text{all chairs}\}$ and $P = \{\text{all people}\}$

$S(c,p)$ represents “p sitting in c”

Negations of Multiply Quantified Statements

- $\sim(\forall c \in C \exists p \in P, S(c,p))$
- $\exists c \in C \forall p \in P, \sim S(c,p)$

where $C = \{\text{all chairs}\}$

and $P = \{\text{all people}\}$

$S(c,p)$ represents “p sitting in c”

Other Variations

- **Exactly one child attends school**

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- **Exactly one child attends school**
 - $\exists c \in C \exists s \in S A(c,s) \wedge \sim [\exists p \in C \exists b \in S, p \neq c \wedge A(p,b)]$
 - $\exists c \in C \exists s \in S, A(c,s) \wedge [\forall p \in C \forall b \in S, p = c \vee \sim A(p,b)]$

Other Variations

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- **At most 1 child attends school**

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- **At most 1 child attends school**
 - $\forall c,p \in C \forall s,b \in S, (A(c,s) \wedge A(p,b)) \rightarrow c=p$

Other Variations

- **Exactly one child attends school**
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- **At most 1 child attends school**
 - $\forall c,p \in C \forall s,b \in S, (A(c,s) \wedge A(p,b)) \rightarrow c=p$
- **At least 2 children attend school**

Other Variations

- **Exactly one child attends school**
 - $\exists c \in C \exists s \in S A(c,s) \wedge \sim [\exists p \in C \exists b \in S, p \neq c \wedge A(p,b)]$
 - $\exists c \in C \exists s \in S, A(c,s) \wedge [\forall p \in C \forall b \in S, p = c \vee \sim A(p,b)]$
- **At most 1 child attends school**
 - $\forall c,p \in C \forall s,b \in S, (A(c,s) \wedge A(p,b)) \rightarrow c=p$
- **At least 2 children attend school**
 - $\exists c,p \in C \exists s,b \in S, A(c,s) \wedge A(p,b) \wedge p \neq c$

Degenerate or Vacuous Cases

- $\forall s \mathbf{B}(s)$ - all my students are wearing blue
 - $B(s)$ "student s is wearing blue"
- $\forall s \forall c \mathbf{I}(s,c)$
- $\forall s \exists c \mathbf{I}(s,c)$
- $\exists c \forall s \mathbf{I}(s,c)$
 - $I(s,c)$ "student s is in class c "

If there are no students...

Variants of Quantified Conditional Statements

- **Statement:** $\forall x \in D, P(x) \rightarrow Q(x)$
- **Contrapositive:** $\forall x \in D, \sim Q(x) \rightarrow \sim P(x)$
- **Converse:** $\forall x \in D, Q(x) \rightarrow P(x)$
- **Inverse:** $\forall x \in D, \sim P(x) \rightarrow \sim Q(x)$
- Also applies to Existentially Quantified Conditional Statements

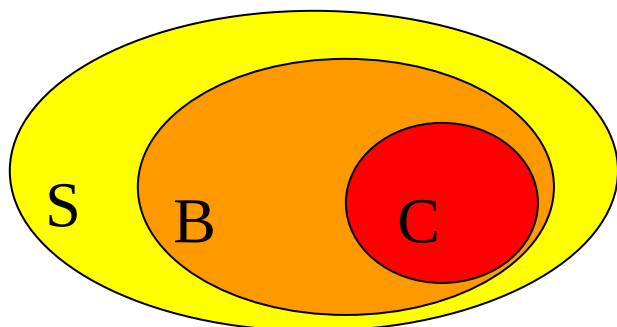
Euler Diagrams

- Circles used to tell "Truth Sets" for the predicate
 - Where the predicate applied to a object is true
- A dot is used to tell a specific instance
- If "all" then a completely contained circle
- If "some" then an overlapping circle

All college students are brilliant.

All brilliant people are scientists.

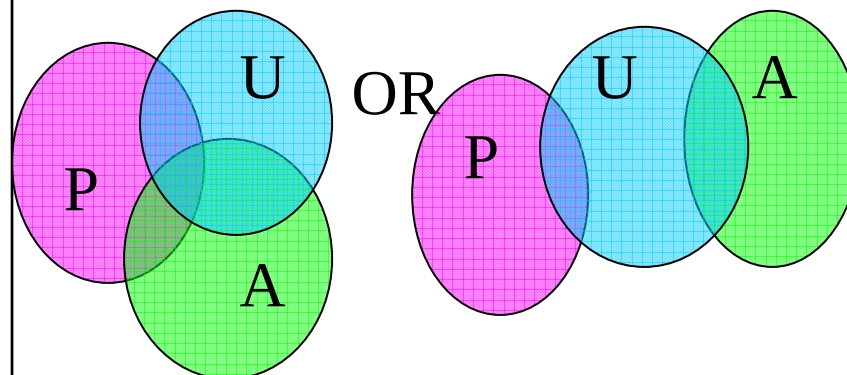
$\therefore$ All college students are scientists.



Some poets are unsuccessful.

Some athletes are unsuccessful.

$\therefore$ Some poets are athletes.



Rules of Inference for Quantified Statements

Universal Modus Ponens $\forall x \in D, P(x) \rightarrow Q(x)$ $P(a)$ $a \in D$ $\therefore Q(a)$	Universal Modus Tollens $\forall x \in D, P(x) \rightarrow Q(x)$ $\sim Q(a)$ $a \in D$ $\therefore \sim P(a)$
Universal Instantiation $\forall x \in D, P(x)$ $a \in D$ $\therefore P(a)$	Existential Generalization $P(c)$ $c \in D$ $\therefore \exists x \in D, P(x)$

Rules that DON'T exist or need more clarification

- Existential Modus Ponens - Doesn't exist
- Existential Modus Tollens - Doesn't exist
- Universal Generalization: $P(a) \therefore \forall x \in D, P(x)$
 - only if a is completely arbitrary in the domain
- Existential Instantiation: $\exists x \in D, P(x) \therefore P(a)$
 - only if a is completely arbitrary in the domain

Errors in Deduction

- Converse Error

$\forall x \in D, P(x) \rightarrow Q(x)$

$Q(a)$

$\therefore P(a)$

- Called: Asserting the Consequence

- Inverse Error

$\forall x \in D, P(x) \rightarrow Q(x)$

$\sim P(a)$

$\therefore \sim Q(a)$

- Called: Denying the Hypothesis

Easy Formal Direct Proofs by Deduction

- $\forall x \in D, P(x) \rightarrow Q(x)$
- $\sim Q(a)$ where $a \in D$
- therefore: $\exists x \in D, \sim P(x)$
- -----
- $\forall x \in D, P(x) \rightarrow Q(x)$
- $\forall x \in D, R(x) \rightarrow \sim P(x)$
- $P(b)$ where $b \in D$
- therefore: $Q(b) \wedge \sim R(b)$

More Formal Direct Proofs by Deduction

- $\forall x \in D, P(x) \rightarrow Q(x)$
- $\forall x \in D, \sim P(x) \vee R(x)$
- $P(b)$ where $b \in D$
- therefore: $\exists x \in D, Q(x) \wedge R(x)$

=====

- $\forall x \in D, P(x) \rightarrow Q(x)$
- $\forall y \in D, \sim P(y) \rightarrow R(y)$
- $\exists z \in D, \sim Q(z)$
- therefore: $\exists x \in D, R(x)$

$A(c,s)$ = "child c attends school s "

- $\exists c \exists s A(c,s)$
 - find one child/school combo which makes it true
 - one child attends some school somewhere
- $\forall c \forall s A(c,s)$
 - must be true for all child/school combos
 - all children must attend all schools
- $\forall c \exists s A(c,s)$
 - for all children select any one school to which that child attends
 - all children attend some school
- $\forall s \exists c A(c,s)$
 - for all schools select any one child to which that school attends
 - all schools have at least one child
- $\exists s \forall c A(c,s)$
 - select any one school and assert that all children attend that one school
 - there is a school that all children attend
- $\exists c \forall s A(c,s)$
 - select any one child and assert that all schools are attended by that one child
 - there is a child who attends all schools

$A(c,s)$ = "child c attends school s "

- Negation of “Every child attends school”.
 - $\sim[\forall c \exists s A(c,s)]$
- At least one child did not attend school.

$A(c,s)$ = "child c attends school s "

- Negation of "Every child attends school".
 - $\sim[\forall c \exists s A(c,s)]$
- At least one child did not attend school.
 - $\sim[\forall c \exists s A(c,s)]$
 - It is not the case that all children attend school.
 - $\exists c \sim[\exists s A(c,s)]$
 - There is one child for whom it is not the case that there exists a school which he/she attends.
 - $\exists c \forall s \sim A(c,s)$
 - There is one child for whom all schools are ones that he/she does not attend.

$A(c,s)$ = "child c attends school s "

- Negation of “At least one child attends school”.
 - $\sim [\exists c \exists s A(c,s)]$
- No children attends school.

$A(c,s)$ = "child c attends school s "

- Negation of "At least one child attends school".
 - $\sim [\exists c \exists s A(c,s)]$
- No children attend school.
 - $\sim [\exists c \exists s A(c,s)]$
 - It is not the case that there is a child who attends school.
 - $\forall c \sim [\exists s A(c,s)]$
 - For all children it is not the case that you can select a school that child attends.
 - $\forall c \forall s \sim A(c,s)$
 - For all child/school combinations it is not the case that the child attends that school.

More Practice in Translation

- No two people share the same toothbrush.

More Practice in Translation

- No two people share the same toothbrush.
- Development:

It is not the case (there there are two (or more) people sharing a toothbrush).

$\sim(\text{there are two (or more) people sharing a toothbrush})$

$\sim(\text{at least two people share a toothbrush})$

$$\sim(\exists s, m \in P \exists t \in T, K(s, t) \wedge K(m, t) \wedge s \neq m)$$

Another way:

If we look at every pair of people/toothbrush combination, one of the three is false.

$$\forall s, m \in P \forall t \in T, \sim K(s, t) \vee \sim K(m, t) \vee s = m$$

More Practice in Translation

- There is a person only a mom could love.

More Practice in Translation

- There is a person only a mom could love.
- Development:

There is at least one person (only a mom could love).

There is at least one person (if anyone loves him it must be a mom)

There is at least one person (if anyone loves him then that person is a mom)

$$\exists \mathbf{x} \in \mathbf{P} \forall \mathbf{s} \in \mathbf{P}, \mathbf{L(s,x)} \rightarrow \mathbf{M(s)}$$

$\mathbf{L(s,x)}$ means "s loves x"

$\mathbf{M(s)}$ means "s is a mom"

Another way:

There is at least one person, x, (There's nobody in the world who loves x is not a Mom)

There's a person, x, (it's not the case (there is a person who Loves x and is not a Mom)).

$$\exists \mathbf{x} \in \mathbf{P} [\sim (\exists \mathbf{s} \in \mathbf{P}, \mathbf{L(s,x)} \wedge \sim \mathbf{M(s)})]$$

One More Proof

P1 $\forall x \in D, [A(x) \vee B(x)] \rightarrow [M(x) \vee N(x)]$

P2 $\exists y \in D, A(y) \wedge \sim N(y)$

therefore $\exists z \in D, M(z) \vee B(z)$