Predicate Calculus

- Subject / Predicate

John / went to the store.

The sky / is blue.

- Propositional Logic - uses statements
- Predicate Calculus - uses predicates
  - predicates must be applied to a subject in order to be true or false
- P(x)
  - means this predicate represented by P
  - applied to the object represented by x

Quantification

- $\exists x$ There exists an x
- $\forall x$ For all x's

Usually specified from a domain

- $\exists x \in Z$ There exists an x in the integers
- $\forall x \in R$ For all x's in the reals
- Domain - set where these subjects come from

Translation

- **A student of mine is wearing a blue shirt.**
  - Domain: people who are my students S
  - Quantification: There is at least one
  - Predicate: wearing a blue shirt

  $\exists x \in S$ such that B(x)

  where B(x) represents "wearing a blue shirt"
- **My students are in class.**
  - Domain: people who are my students S
  - Quantification: All of them
  - Predicate: are in class

  $\forall x \in S$ such that C(x)

  where C(x) represents "being in class"

Negation of Quantified Statements

~ ($\exists$x $\in$ people such that **H(x))**

$\equiv$ **$\forall$x $\in$ people** such that **~ H(x)**

~(There is a person who is here.) $\equiv$

For all people, each person is not here.

same in meaning as "There is no person here."

~ ($\forall$ x $\in$ people such that **H(x))**

$\equiv$ **$\exists$ x $\in$ people** such that **~ H(x)**

~(For all people, each person is here.) $\equiv$

There is at least one person who is not here.

Multiple Predicate Translation

- **A student of mine is wearing a blue shirt.**
  - Domain: all people P
  - Quantification: There is at least one
  - Predicates: "wearing a blue shirt" and "is my student"

  $\exists$x $\in$ P such that B(x) ^ S(x)

  B(x) represents "wearing a blue shirt"

  S(x) represents "being my student"
- **My students are in class.**
  - Domain: all people P
  - Quantification: All of them
  - Predicates: "are in class" and "is my student"

  $\forall$x $\in$ P such that S(x) $\rightarrow$ C(x)

  C(x) represents "being in class"

  S(x) represents "being my student"

Multiple Quantification

- **$\exists p \in P$ $\exists c \in C$, S(c,p)**
- **$\exists c \in C$ $\exists p \in P$, S(c,p)**

- **$\forall p \in P$ $\forall c \in C$, S(c,p)**
- **$\forall c \in C$ $\forall p \in P$, S(c,p)**

where **C ={all chairs}** and **P ={all people}**

and **S(c,p) represents “p sitting in c"**

Mixed Multiple Quantification

- $\forall c \in C\ \exists p \in P,\ S(c,p)$
- $\forall p \in P\ \exists c \in C,\ S(c,p)$
- $\exists p \in P\ \forall c \in C,\ S(c,p)$
- $\exists c \in C\ \forall p \in P,\ S(c,p)$

where **C = {all chairs}** and **P={all people}**
S(c,p) represents **"p sitting in c"**

Negations of Multiply Quantified Statements

- $\sim(\forall c \in C\ \exists p \in P,\ S(c,p))$
- $\exists c \in C\ \forall p \in P,\ \sim S(c,p)$

  where **C = {all chairs}**
  and **P = {all people}**
  S(c,p) represents **"p sitting in c"**

Other Variations

- **Exactly one child attends school**

Other Variations

- **Exactly one child attends school**
  - $\exists c \in C\ \exists s \in S\ A(c,s) \wedge \sim[\exists p \in C\ \exists b \in S,\ p \neq c \wedge A(p,b)]$
  - $\exists c \in C\ \exists s \in S,\ A(c,s) \wedge [\forall p \in C\ \forall b \in S,\ p=c \vee \sim A(p,b)]$

Other Variations

- **Exactly one child attends school**
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  - $\exists c \in C\ \exists s \in S,\ A(c,s) \wedge [\forall p \in C\ \forall b \in S,\ p=c \vee \sim A(p,b)]$
- **At most 1 child attends school**

Other Variations

- **Exactly one child attends school**
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  - $\exists c \in C\ \exists s \in S,\ A(c,s) \wedge [\forall p \in C\ \forall b \in S,\ p=c \vee \sim A(p,b)]$
- **At most 1 child attends school**
  - $\forall c,p \in C\ \forall s,b \in S,\ (A(c,s) \wedge A(p,b)) \rightarrow c=p$

Other Variations

- **Exactly one child attends school**
  - **$\exists c\in C\ \ \exists s\in S\ A(c,s) \wedge \sim[\exists p\in C\ \exists b\in S,\ \ p\neq c \wedge A(p,b)]$**
  - **$\exists c\in C\ \exists s\in S, A(c,s)\wedge[\forall p\in C\ \forall b\in S,\ \ p=c \vee \sim A(p,b)]$**
- **At most 1 child attends school**
  - **$\forall c,p\in C\ \forall\ s,b\in S, (A(c,s) \wedge A(p,b)) \rightarrow c=p$**
- **At least 2 children attend school**

Other Variations

- **Exactly one child attends school**
  - **$\exists c\in C\ \ \exists s\in S\ A(c,s) \wedge \sim[\exists p\in C\ \exists b\in S,\ \ p\neq c \wedge A(p,b)]$**
  - **$\exists c\in C\ \exists s\in S, A(c,s)\wedge[\forall p\in C\ \forall b\in S,\ \ p=c \vee \sim A(p,b)]$**
- **At most 1 child attends school**
  - **$\forall c,p\in C\ \forall\ s,b\in S, (A(c,s) \wedge A(p,b)) \rightarrow c=p$**
- **At least 2 children attend school**
  - **$\exists c,p\in C\ \exists s,b\in S,\ \ A(c,s) \wedge A(p,b) \wedge p \neq c$**

Degenerate or Vacuous Cases

- **$\forall$s B(s)** - all my students are wearing blue
  - B(s) "student s is wearing blue"
- **$\forall$s $\forall$c I(s,c)**
- **$\forall$s $\exists$c I(s,c)**
- **$\exists$c $\forall$s I(s,c)**
  - I(s,c) "student s is in class c"

If there are no students…

Variants of Quantified Conditional Statements

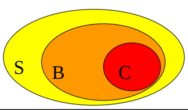
- **Statement:** $\forall \mathbf{x} \in D, P(x) \rightarrow Q(x)$
- **Contrapositive:** $\forall \mathbf{x} \in D, \sim Q(x) \rightarrow \sim P(x)$
- **Converse:** $\forall \mathbf{x} \in D, Q(x) \rightarrow P(x)$
- **Inverse:** $\forall \mathbf{x} \in D, \sim P(x) \rightarrow \sim Q(x)$

- Also applies to Existentially Quantified Conditional Statements

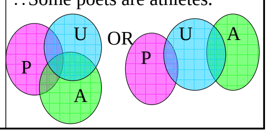
Euler Diagrams

- Circles used to tell "Truth Sets" for the predicate
  - Where the predicate applied to a object is true
- A dot is used to tell a specific instance
- If "all" then a completely contained circle
- If "some" then an overlapping circle

All college students are brilliant.
All brilliant people are scientists.
$\therefore$ All college students are scientists.



Some poets are unsuccessful.
Some athletes are unsuccessful.
$\therefore$ Some poets are athletes.



Rules of Inference for Quantified Statements

Universal Modus Ponens	**Universal Modus Tollens**
$\forall \mathbf{x} \in D, P(x) \rightarrow Q(x)$	$\forall \mathbf{x} \in D, P(x) \rightarrow Q(x)$
$P(a)$	$\sim Q(a)$
$a \in D$	$a \in D$
$\therefore\ Q(a)$	$\therefore\ \sim P(a)$
Universal Instantiation	**Existential Generalization**
$\forall x \in D, P(x)$	$P(c)$
$a \in D$	$c \in D$
$\therefore\ P(a)$	$\therefore\ \exists x \in D, P(x)$

Rules that DON'T exist or need more clarification

- Existential Modus Ponens - Doesn't exist
- Existential Modus Tollens - Doesn't exist

- Universal Generalization: $P(a) \therefore \forall x \in D, P(x)$
  - only if a is completely arbitrary in the domain
- Existential Instantiation: $\exists x \in D, P(x) \therefore P(a)$
  - only if a is completely arbitrary in the domain

Errors in Deduction

- Converse Error

$\forall x \in D, P(x) \rightarrow Q(x)$

$Q(a)$

$\therefore$ P(a)

- Called: Asserting the Consequence

- Inverse Error

$\forall x \in D, P(x) \rightarrow Q(x)$

$\sim P(a)$

$\therefore \sim Q(a)$

- Called: Denying the Hypothesis

Easy Formal Direct Proofs by Deduction

- $\forall x \in D, P(x) \rightarrow Q(x)$
- $\sim Q(a)$ where $a \in D$
- therefore: $\exists x \in D, \sim P(x)$
- ------------------------------------
- $\forall x \in D, P(x) \rightarrow Q(x)$
- $\forall x \in D, R(x) \rightarrow \sim P(x)$
- P(b) where $b \in D$
- therefore: $Q(b) \wedge \sim R(b)$

More Formal Direct Proofs by Deduction

- $\forall x \in D, P(x) \rightarrow Q(x)$
- $\forall x \in D, \sim P(x) \vee R(x)$
- P(b) where $b \in D$
- therefore: $\exists x \in D, Q(x) \wedge R(x)$

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- $\forall x \in D, P(x) \rightarrow Q(x)$
- $\forall y \in D, \sim P(y) \rightarrow R(y)$
- $\exists z \in D, \sim Q(z)$
- therefore: $\exists x \in D, R(x)$

A(c,s) = "child c attends school s"

- **∃ c ∃s A(c,s)**
  - find one child/school combo which makes it true
  - one child attends some school somewhere
- **∀c ∀ s A(c,s)**
  - must be true for all child/school combos
  - all children must attend all schools
- **∀c ∃s A(c,s)**
  - for all children select any one school to which that child attends
  - all children attend some school
- **∀s ∃c A(c,s)**
  - for all schools select any one child to which that school attends
  - all schools have at least one child
- **∃s ∀c A(c,s)**
  - select any one school and assert that all children attend that one school
  - there is a school that all children attend
- **∃c∀s A(c,s)**
  - select any one child and assert that all schools are attended by that one child
  - there is a child who attends all schools

A(c,s) = "child c attends school s"

- Negation of “Every child attends school”.
  - **~[∀c ∃s A(c,s)]**
- At least one child did not attend school.

A(c,s) = "child c attends school s"

- Negation of “Every child attends school”.
  - **$\sim[\forall c\ \exists s\ A(c,s)]$**
- At least one child did not attend school.
  - **$\sim[\forall c\ \exists s\ A(c,s)]$**
    - It is not the case that all children attend school.
  - **$\exists c \sim[\exists s\ A(c,s)]$**
    - There is one child for whom it is not the case that there exists a school which he/she attends.
  - **$\exists c\ \forall\ s \sim A(c,s)$**
    - There is one child for whom all schools are ones that he/she does not attend.

A(c,s) = "child c attends school s"

- Negation of “At least one child attends school”.
  - **$\sim[\exists c\ \exists s\ A(c,s)]$**
- No children attends school.

A(c,s) = "child c attends school s"

- Negation of “At least one child attends school”.
  - **$\sim[\exists c\ \exists s\ A(c,s)]$**
- No children attend school.
  - **$\sim[\exists c\ \exists s\ A(c,s)]$**
    - It is not the case that there is a child who attends school.
  - **$\forall c \sim[\exists s\ A(c,s)]$**
    - For all children it is not the case that you can select a school that child attends.
  - **$\forall c\ \forall s \sim A(c,s)$**
    - For all child/school combinations it is not the case that the child attends that school.

More Practice in Translation

- No two people share the same toothbrush.

More Practice in Translation

- No two people share the same toothbrush.
- Development:

It is not the case (there there are two (or more) people sharing a toothbrush).
~(there are two (or more) people sharing a toothbrush)
~(at least two people share a toothbrush)

$\sim(\exists s,m \in P\ \exists t \in T,\ K(s,t) \wedge K(m,t) \wedge s \neq m)$

Another way:
If we look at every pair of people/toothbrush combination, one of the three is false.

$\forall s,m \in P\ \forall t \in T,\ \sim K(s,t) \vee \sim K(m,t) \vee s=m$

More Practice in Translation

- There is a person only a mom could love.

More Practice in Translation

- There is a person only a mom could love.
- Development:

There is at least one person (only a mom could love).
There is at least one person (if anyone loves him it must be a mom)
There is at least one person (if anyone loves him then that person is a mom)

$\exists x \in P\ \forall s \in P,\ L(s,x) \rightarrow M(s)$

L(s,x) means "s loves x"
M(s) means "s is a mom"

Another way:
There is at least one person, x,(There's nobody in the world who loves x is not a Mom)
There's a person,x, (it's not the case (there is a person who Loves x and is not a Mom)).

$\exists x \in P[\sim(\exists\ s \in P,\ L(s,x)\ \wedge \sim M(s))]$

One More Proof

P1 $\forall x \in$ **D, [A(x) v B(x)] $\rightarrow$ [M(x) v N(x)]**

P2 $\exists$**y$\in$D, A(y) $\wedge$ ~N(y)**

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therefore $\exists$z$\in$**D, M(z) v B(z)**