Sets

Definition of a Set:

NAME = {list of elements or description of elements}

i.e. $B = \{1,2,3\}$ or $C = \{x \in Z^+ \mid -4 < x < 4\}$

Axiom of Extension:

A set is completely defined by its elements

i.e. $\{a,b\} = \{b,a\} = \{a,b,a\} = \{a,a,a,b,b,b\}$

Subset

$A \subseteq B \leftrightarrow \forall x \in U, x \in A \rightarrow x \in B$

A is contained in B

B contains A

$A \nsubseteq B \leftrightarrow \exists x \in U, x \in A \ \wedge x \notin B$

Relationship between membership and subset:

$\forall x \in U, x \in A \leftrightarrow \{x\} \subseteq A$

Definition of set equality: $A = B \leftrightarrow A \subseteq B \wedge B \subseteq A$

Same Set or Not??

$X=\{x \in Z \mid \exists p \in Z, x = 2p\}$

$Y=\{y \in Z \mid \exists q \in Z, y = 2q-2\}$

$A=\{x \in Z \mid \exists i \in Z, x = 2i+1\}$

$B=\{x \in Z \mid \exists i \in Z, x = 3i+1\}$

$C=\{x \in Z \mid \exists i \in Z, x = 4i+1\}$

Set Operations

<u>Formal Definitions and Venn Diagrams</u>

Union:

$$A \bigcup B = \{x \in U \mid x \in A \vee x \in B\}$$

Intersection:

$$A \bigcap B = \{x \in U \mid x \in A \wedge x \in B\}$$

Complement:

$$A^c = A' = \{x \in U \mid x \notin A\}$$

Difference:

$$A - B = \{x \in U \mid x \in A \wedge x \notin B\}$$

$$A - B = A \bigcap B'$$

Ordered n-tuple and the Cartesian Product

- Ordered n-tuple – takes order and multiplicity into account
- $(x_1,x_2,x_3,\ldots,x_n)$
  - n values
  - not necessarily distinct
  - in the order given
- $(x_1,x_2,x_3,\ldots,x_n) = (y_1,y_2,y_3,\ldots,y_n) \leftrightarrow \forall i \in Z^{1\le i\le n}, x_i=y_i$

- Cartesian Product

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

Formal Languages

- Σ = alphabet = a finite set of symbols
- string over Σ =

  empty (or null) string denoted as ε

  OR

  ordered n-tuple of elements
- Σ^n = set of strings of length n
- Σ^* = set of all finite length strings

Empty Set Properties

1. Ø is a subset of every set.
2. There is only one empty set.
3. The union of any set with Ø is that set.
4. The intersection of any set with its own complement is Ø.
5. The intersection of any set with Ø is Ø.
6. The Cartesian Product of any set with Ø is Ø.
7. The complement of the universal set is Ø and the complement of the empty set is the universal set.

Other Definitions

- Proper Subset

$$A \subset B \leftrightarrow A \subseteq B \wedge A \neq B$$

- Disjoint Set

A and B are disjoint

$\leftrightarrow$ A and B have no elements in common

$\leftrightarrow \forall x \in U,\ x \in A \rightarrow x \notin B \wedge x \in B \rightarrow x \notin A$

$A \cap B = \emptyset \leftrightarrow$ A and B are Disjoint Sets

- Power Set

$\mathcal{P}(A)$ = set of all subsets of A

Properties of Sets in Theorems 5.2.1 & 5.2.2

- Inclusion $A \cap B \subseteq A \qquad A \cap B \subseteq B$

  $A \subseteq A \cup B \qquad B \subseteq A \cup B$

- Transitivity

$$A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$$

- DeMorgan's for Complement

$$(A \cup B)' = A' \cap B' \qquad (A \cap B)' = A' \cup B'$$

- Distribution of union and intersection

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Using Venn Diagrams to help find counter example

$$A \bigcup (B \bigcap C) = ? = (A \bigcap B) \bigcup (A \bigcap C)$$

$$A \bigcup (B \bigcap C) = ? = (A \bigcap B) \bigcap C$$

Deriving new Properties using rules and Venn diagrams

$$B-(A\bigcap C)=(B-A)\bigcup(B-C)$$

$$A-B=A-(A\bigcap B)$$

$$A\subseteq B\wedge A\subseteq C\rightarrow A\subseteq(B\bigcap C)$$

Partitions of a set

- A collection of nonempty sets $\{A_1, A_2, \ldots, A_n\}$ is a partition of the set A

if and only if

1. $A = A_1 \cup A_2 \cup \ldots \cup A_n$
2. $A_1, A_2, \ldots, A_n$ are mutually disjoint

Proofs about Power Sets

Power set of A = $\mathcal{P}$(A) = Set of all subsets of A

- Prove that

$$\forall \mathbf{A,B} \in \{\mathbf{sets}\}, \mathbf{A} \subseteq \mathbf{B} \rightarrow \mathcal{P}(\mathbf{A}) \subseteq \mathcal{P}(\mathbf{B})$$

- Prove that (where n(X) means the size of set X)

$$\forall \mathbf{A} \in \{\mathbf{sets}\}, \mathbf{n(A) = k} \rightarrow \mathbf{n}(\mathcal{P}(\mathbf{A})) = \mathbf{2^k}$$