

Counting Elements in a List

- How many integers in the list from 1 to 10?
- How many integers in the list from m to n ?
(assuming $m \leq n$)

How many in a list divisible by something:

- How many positive three digit integers are there?
 - (this means only the ones that require 3 digits)
 - $999 - 100 + 1 = 900$

- How many three digit integers are divisible by 5?
 - think about the definition of divisible by

$x \mid y \leftrightarrow \exists k \in \mathbb{Z}, y = kx$ and then count the k 's that work

100, 101, 102, 103, 104, 105, 106, ... 994, 995, 996, 997, 998, 999

↑
20*5

↑
21*5

...

↑
199*5

- count the integers between 20 and 199
- $199 - 20 + 1 = 180$

Probability

likelihood of a specific event

- Sample Space = set of all possible outcomes
- Event = subset of sample space
- Equal Probability Formula:
 - Given a finite sample space S where all outcomes are equally likely
 - Select an event E from the sample space S
 - The probability of event E from sample space S :

$$P(E) = \frac{n(E)}{n(S)}$$

Flipping Two Coins

- Sample Space = $\{(H,H), (H,T), (T,H), (T,T)\}$
- Probability of no heads
- Probability of at least one head
- Probability of same sides on the two coins
- Note: probability & actual outcomes often differ

Standard Playing Cards

- values: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- suits: D(♦), H(♥), S(♠), C(♣)
- probability of drawing the Ace of Spades
- probability of drawing a Spade
- probability of drawing a face card
- probability of drawing a red face card

Rolling Two Six-Sided Dice

- Sample Space
 $\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),$
 $(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),$
 $\dots$
 $(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$
- Probability of rolling a 10
- Probability of rolling a pair

Multi-level Probability

- If I toss a coin once – the probability of Head = $\frac{1}{2}$
- If I toss that coin 5 times
 - the probability of all heads

$$\frac{1}{2} * \frac{1}{2} * \frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{1}{2^5}$$

- the probability of exactly 4 heads

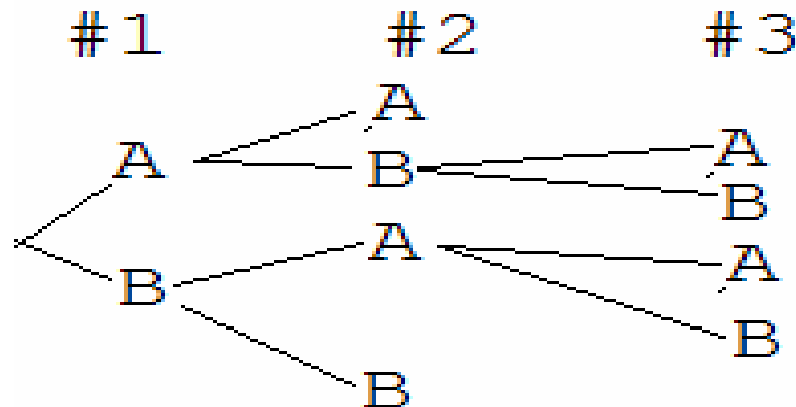
$$\frac{5}{2^5}$$

Multiplication Rule

- 1st step can be performed n_1 ways
 - 2nd step can be performed n_2 ways
 - ...
 - K^{th} step can be performed n_k ways
-
- operation can be performed $n_1 * n_2 * \dots * n_k$ ways
-
- Cartesian product $n(A)=3, n(B)=2, n(C)=4$
 - $n(A \times B \times C) = 24$
 - $n(A \times B) = 6 \quad n((A \times B) \times C) = 24$

Tournament Play

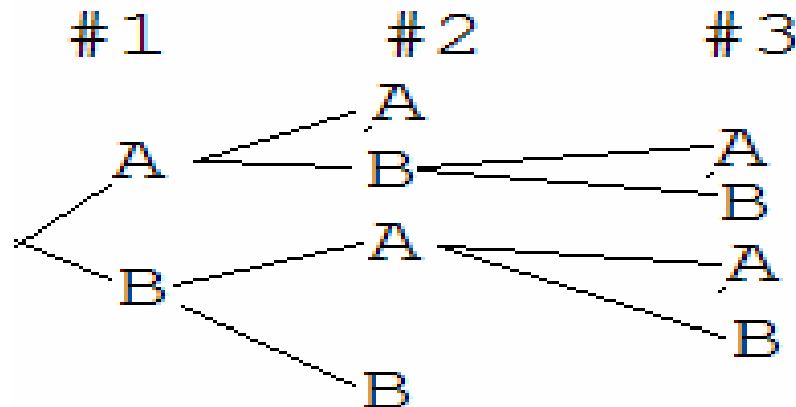
- Team A and Team B in “Best of 3” Tournament
- where they each have an equal likelihood of winning each game



- Do leaves add up to 1?
- Do we have to play 3 games?
- Do A and B have an equal chance of winning?

What if A wins 2 of every 3 games?

- Each line for A must have a $2/3$
- Each line for B must have a $1/3$



- How likely is A to win the tournament?
- How likely is B to win the tournament?

Using the Multiplication Rule for Selecting a PIN

- Number of 4 digit PINs of (0,1,2,..)
 - with repetition allowed = $4*4*4*4=256$
 - with no repetition allowed = $4*3*2*1=24$
- Extra Rules :
 - . (the period) can't be first or last
 - 0 can't be first
 - with repetition allowed = $2*4*4*3$
 - without repetition allowed = $2*2*2*1$

Probabilities with PINs

- Number of 4 digit PINs of (0,1,2,..)
 - with repetition allowed = $4*4*4*4=256$
 - with no repetition allowed = $4*3*2*1=24$
- What is the probability that your 4 digit PIN has no repeated characters?
- What is the probability that your 4 digit PIN does have repeated characters?
- probability of the complement of an event
$$P(E') = P(E^c) = 1-P(E)$$

Difference Rule Formally

- If A is a finite set and $B \subseteq A$, then

$$n(A-B) = n(A) - n(B)$$

- One Application:

probability of the complement of an event

$$P(E') = P(E^c) = 1 - P(E)$$

PINs with less specified length

Addition Rule

- Assume it can be a 2,3 or 4 length PIN

Partition the problem

number of 2 length PINs w/rep allowed: $4*4 = 16$

number of 3 length PINs w/rep allowed: $4*4*4 = 64$

number of 4 length PINs w/rep allowed: $4*4*4*4 = 256$

Number PINs if allowing length of 2,3 or 4 = 336

Addition Rule Formally

- if $A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k = A$
- and $A_1, A_2, A_3, \dots, A_k$ are pairwise disjoint

in other words, if these subsets form a partition of A

$$n(A) = n(A_1) + n(A_2) + n(A_3) + \dots + n(A_k)$$

Another example for Multiplication Rule and Addition Rule

- How many 3 digit integers are divisible by 5?
 - How many end in a 0? $9*10*1 = 90$
 - How many end in a 5? $9*10*1 = 90$
 - These form a partition with the set of numbers divisible by 5 so
 - $90 + 90 = 180$

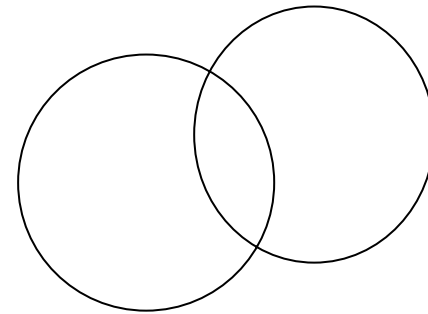
Where Multiplication Rule Doesn't Work

- People= {Angel, Bob, Carol, Dan}
- need to be appointed as
 - president, vice-president, and treasurer
 - nobody can hold more than one office
 - Angel doesn't want to be president
 - Only Bob and Dan want to be vice-president

Inclusion/Exclusion Rule

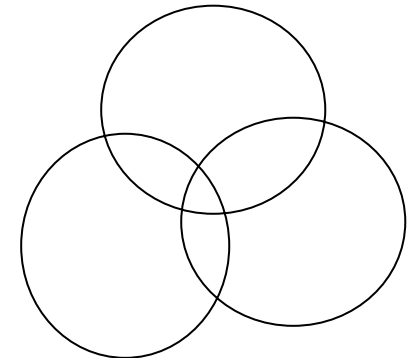
- If there are two sets:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$



- If there are three sets:

$$\begin{aligned} n(A \cup B \cup C) = & n(A) + n(B) + n(C) \\ & - n(A \cap B) - n(A \cap C) - n(B \cap C) \\ & + n(A \cap B \cap C) \end{aligned}$$



Permutations

- Different ways of arranging objects
 - in a line or circle
 - without duplication/ all items distinguishable
 - note: order is taken into account
- Number of linear permutations of N objects = $N!$
 N possible for 1st position * $(N-1)$ for 2nd * ... * (1) for last
- Number of circular permutations of N objects = $(N-1)!$
Fix one person,
then $(N-1)$ possible for next position * $(N-2)$ for 2nd * ... * (1) for last

r-Permutations

If there are n things in the set,
and you want to line-up only r of them.

$$P(n, r) = {}_r P_n = \frac{n!}{(n - r)!}$$

- Example:

Class = {Alice, Bob, Carol, Dan}

– select a president and a vice president to represent the class

Combinations

- Different ways of selecting objects
 - Counting Subsets
 - without duplication/ all items distinguishable
 - note: order is not taken into account

$$\forall n, r \in \mathbb{Z}^{\geq 0} \text{ where } n \geq r,$$

$$C(n, r) = \binom{n}{r} = \frac{P(n, r)}{r!} = \frac{n!}{(n-r)!r!}$$

- Examples:

Class = {Alice, Bob, Carol, Dan}

- select two class representatives
- select three class representatives

Harder Examples

selecting “class representatives”

Class = {Alice,Bob,Carol,Dan, Erin, Fred}

- select 2 – no restrictions
- select 2 – assuming Alice and Bob must stay together
- select 3 – no restrictions
- select 3 – assuming Alice and Bob must stay together
- select 3 – assuming Alice and Bob refuse to serve together

Different Types of Members

{Alice, Bob, Carol, Dan, Erin, Fred, George, Harry}

pink names are girls and blue names are boys

- 8 people in the set: 3 girls & 5 boys
- make a 5 member team of 2 girls and 3 boys
- make a 5 member team that has only one girl
- make a 5 member team that has no girls
- make a 5 member team that has at least one girl

Permutations but of indistinguishable items

- Assume you have a set of 15 beads
 - 3 Red
 - 2 Black
 - 4 orange
 - 6 green
- Select positions of R's, then B's, then O's then G's

$$\binom{15}{3} * \binom{12}{2} * \binom{10}{4} * \binom{6}{6} = \frac{15!}{3!2!4!6!}$$

Combinations with Repetition

- $\{a,b,c,d,e\}$
- How many 3-combinations can I make without repetition?
- How many 3-combinations can I make with unlimited repetition allowed?
- these are multisets $[a,b,c]$
 - not sets $\{a,b,c\}$
 - not tuples (a,b,c)

Probability with Combinations

- Assume there are 32 people in the class
- And that 7 will be chosen to get extra homework
- What is the probability that you get extra homework
- Number of ways to select the “lucky 7”
- Number of ways to select if “I get HW”
- $P(\text{I get HW})$

Properties of r-Permutations and proofs of those properties

$$P(n,1) = n$$

$$P(n,2) = n^2 - n$$

$$P(n,2) + P(n,1) = n^2$$

$$P(n,n) = n!$$

$$P(n,n-1) = n!$$

Properties of Combinations and their proofs

$\binom{n}{0} = 1$	$\binom{n}{n} = 1$
$\binom{n}{1} = n$	$\binom{n}{n-1} = n$
$\binom{n}{2} = \frac{n(n-1)}{2}$	$\binom{n}{n-2} = \frac{n(n-1)}{2}$
$\binom{n}{r} = \binom{n}{n-r}$	

Binomial Theorem

- $(x+y)^2$
- $(x+y)^3$
- ...
- $(x+y)^n$

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n-1}, \binom{n}{n}$$

Notice Similarities

- The number of Non-negative Integer Solutions of the equation

$$x_1 + x_2 + \dots + x_n = r \quad x_i \geq 0, \forall i \in \mathbb{Z}^{1 \leq i \leq n}$$

- The number of selections, with repetition, of size r from a collection of size n .
- The number of ways r identical objects can be distributed among n distinct containers.

Conditional Probability

- Probability of B given that A is known to have happened

$$P(B | A) = \frac{P(A \cap B)}{P(A)}$$

- If $P(B) = P(B|A)$ then
event B is Independent of event A