Terminology

- Domain: set which holds the values to which we apply the function
- Co-domain: set which holds the possible values (results) of the function
- Range: set of actual values received when applying the function to the values of the domain

Function

- A “total” function is a relationship between elements of the domain and elements of the co-domain where each and every element of the domain relates to one and only one value in the co-domain
- A “partial” function does not need to map every element of the domain.
- f: $X \rightarrow Y$
  - f is the function name
  - X is the domain
  - Y is the co-domain
  - $x \in X$ $y \in Y$ f sends x to y
  - $f(x) = y$ f of x ; value of f at x ; image of x under f

Formal Definitions

- Range of f = $\{y \in Y \mid \exists x \in X, f(x) = y\}$
  - where X is the domain and Y is the co-domain
- Inverse image of y = $\{x \in X \mid f(x) = y\}$
  - the set of things that map to y
- Arrow Diagrams
  - Determining if they are functions using the Arrow Diagram

Teminology of Functions

- Equality of Functions

$\forall f,g \in \{\text{functions}\}$, $f=g \leftrightarrow \forall x \in X,\ f(x)=g(x)$

- F is a One-to-One (or Injective) Function iff

$\forall x_1,x_2 \in X \quad F(x_1) = F(x_2) \rightarrow x_1=x_2$

$\forall x_1,x_2 \in X \quad x_1$ ✎ $x_2 \rightarrow F(x_1)$ ✎ $F(x_2)$

- F is NOT a One-to-One Function iff

$\exists\ x_1,x_2 \in X$, $(F(x_1) = F(x_2))$ ^ $(x_1$ ✎ $x_2)$

- F is an Onto (or Surjective) Function iff

$\forall y \in Y \quad \exists x \in X,\ F(x) = y$

- F is NOT an Onto Function iff

$\exists y \in Y\ \forall x \in X,\ F(x)$ ✎ y

Proving functions one-to-one and onto

$f: R \rightarrow R \qquad f(x)=3x-4$

- Prove or give a counter example that f is one-to-one

use def: $$\forall x_1, x_2 \in R, f(x_1) = f(x_2) \rightarrow x_1 = x_2$$

- Prove or give a counter example that f is onto

use def: $$\forall y \in R \exists x \in R, f(x) = y$$

One-to-One Correspondence or Bijection

F:X $\rightarrow$Y is bijective $\leftrightarrow$ F:X $\rightarrow$Y is one-to-one & onto

- F:X $\rightarrow$Y is bijective $\leftrightarrow$ It has an inverse function

$$\exists F^{-1} : Y \rightarrow X$$

$$\forall x \in X, F(x) = y \rightarrow F^{-1}(y) = x$$

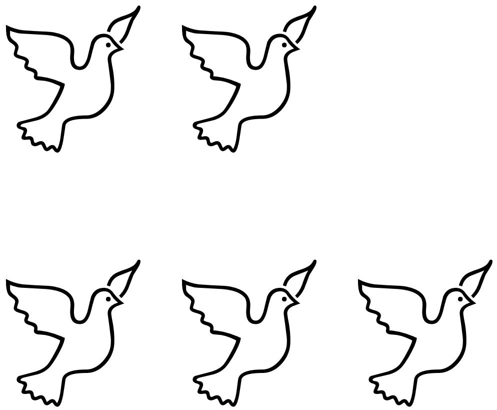
$$\forall y \in Y, F^{-1}(y) = x \rightarrow F(x) = y$$

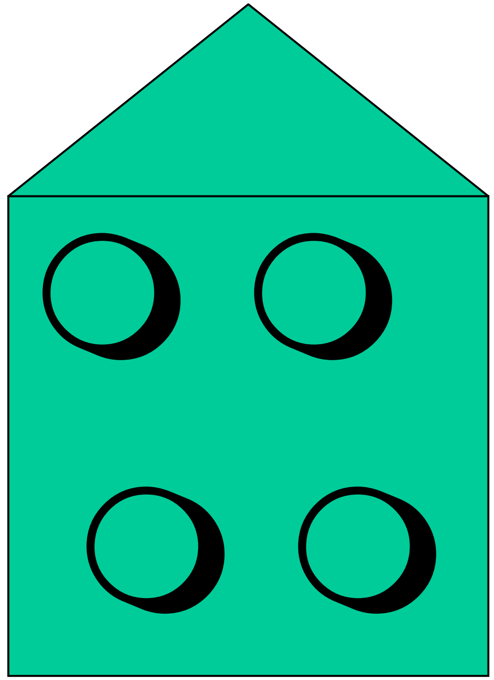
Proving something is a bijection

- $F: Z \rightarrow Z \quad F(x)=x+1$
- Prove it is one-to-one
- Prove it is onto
- Then it is a bijection

- So it has an inverse function
  - find F^{-1}

Pigeonhole Principle





- Basic Form

  A function from one finite set to a smaller finite set cannot be one-to-one; there must be at least two elements in the domain that have the same image in the co-domain.

Examples

- Using this class as the domain,

Must two people share a birthmonth?

Must two people share a birthday?

- A = {1,2,3,4,5,6,7,8}

If I select 5 integers at random from this set, must two of the numbers sum to 9?

If I select 4 integers?

Other (more useful) Forms of the Pigeonhole Principle

- Generalized Pigeonhole Principle
  - For any function f from a finite set X to a finite set Y and for any positive integer k, if $n(X) > k*n(Y)$, then there is some $y \in Y$ such that y is the image of at least $k+1$ distinct elements of X.

- Contrapositive Form of Generalized Pigeonhole Principle
  - For any function f from a finite set X to a finite set Y and for any positive integer k, if for each $y \in Y$, $f^{-1}(y)$ has at most k elements, then X has at most $k*n(Y)$ elements.

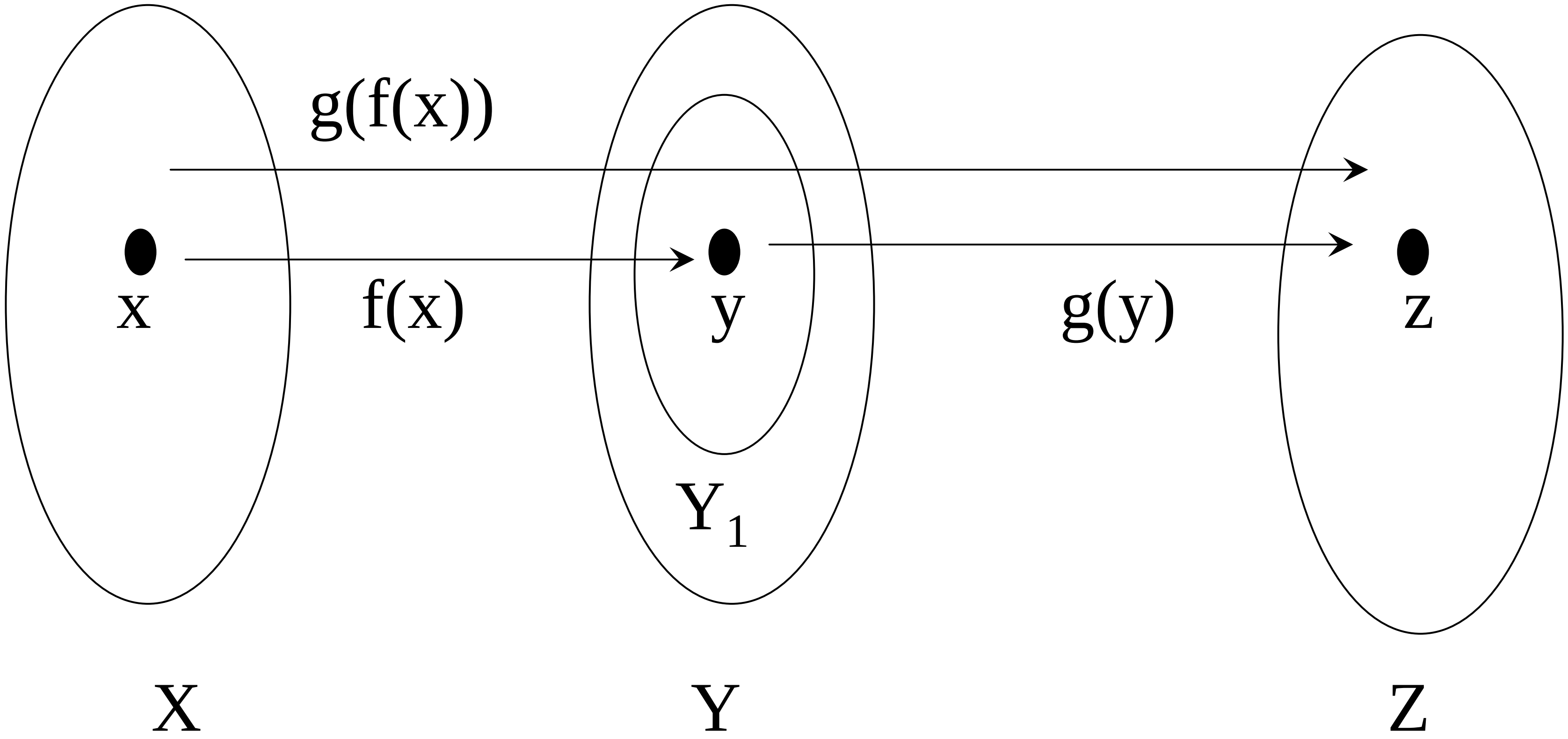
Examples

- Using Generalized Form:
  - Assume 50 people in the room, how many must share the same birthmonth?
  - $n(A)=5$ $n(B)=3$ $F:P(A)\rightarrow P(B)$

    How many elements of $P(A)$ must map to a single element of $P(B)$?
- Using Contrapositive of the Generalized Form:
  - $G:X\rightarrow Y$ where Y is the set of 2 digit integers that do not have distinct digits. Assuming no more than 5 elements of X can map to a single element of Y, how big can X be?
  - You have 5 busses and 100 students. No bus can carry over 25 students. Show that at least 3 busses must have over 15 students each.

Composition of Functions

- $f:X \rightarrow Y_1$ and $g:Y \rightarrow Z$ where $Y_1 \subseteq Y$
  - $g \circ f : X \rightarrow Z$ where $\forall x \in X,\ g(f(x)) = g \circ f(x)$



Composition on finite sets

- Example
  - X = {1,2,3} Y1 = {a,b,c,d} Y={a,b,c,d,e} Z = {x,y,z}

f(1)=c	g(a)=y	$g \circ f(1)=g(f(1))=z$
f(2)=b	g(b)=y	$g \circ f(2)=g(f(2))=y$
f(3)=a	g(c)=z	$g \circ f(3)=g(f(3))=y$
	g(d)=x	
	g(e)=x	

Composition for infinite sets

$f:Z \to Z \quad f(n)=n+1$

$g:Z \to Z \quad g(n)=n^2$

$g \circ f(n)=g(f(n))=g(n+1)=(n+1)^2$

$f \circ g(n)=f(g(n))=f(n^2)=n^2+1$

note: $g \circ f \neq f \circ g$

Identity function

- i_X the identity function for the domain X

  $i_X : X \to X \quad \forall x \in X, i_X(x) = x$

- i_Y the identity function for the domain Y

  $i_Y : Y \to Y \quad \forall y \in Y, i_Y(y) = y$

- composition with the identity functions

Composition of a function with its inverse function

- $f \circ f^{-1} = i_Y$
- $f^{-1} \circ f = i_X$

- Composing a function with its inverse returns you to the starting place.

(note: $f: X \rightarrow Y$ and $f^{-1}: Y \rightarrow X$)

One-to-One in Composition

- If $f:X\rightarrow Y$ and $g:Y\rightarrow Z$ are both one-to-one, then $g \circ f:X\rightarrow Z$ is one-to-one

- If $f:X\rightarrow Y$ and $g:Y\rightarrow Z$ are both onto, then $g \circ f:X\rightarrow Z$ is onto when $Y = Y_1$

Cardinality

- Comparing the "sizes" of sets
  - finite sets ($\emptyset$ or has a bijective function from it to $\{1,2,\ldots,n\}$)
  - infinite sets (can't have a bijective function from it to $\{1,2,\ldots,n\}$)

- $\forall A,B \in \{\text{sets}\}$, A and B have the same cardinality $\leftrightarrow$ there is a one-to-one correspondence from A to B

In other words,

$\text{Card}(A) = \text{Card}(B) \leftrightarrow \exists f \in \{\text{functions}\}, f{:}A \rightarrow B \wedge f$ is a bijection

Countability of sets of integers

- Z^+ is a countably infinite set
- $Z^{\geq 0}$ is a countably infinite set
- Z is a countably infinite set
- Z^{even} is also a countably infinite set

$$\text{Card.}(Z^+)=\text{Card.}(Z^{\geq 0})=\text{Card.}(Z)=\text{Card.}(Z^{even})$$

Real Numbers

- We'll take just a part of this infinite set

- Reals between 0 and 1 (non-inclusive)
  - $X = \{x \in R| \quad 0<x<1\}$

- All elements of X can be written as
  - $0.a_1a_2a_3\ldots\ a_n\ldots$

Cantor's Proof

- Assume the set $X = \{x \in R | 0 < x < 1\}$ is countable
- Then the elements in the set can be listed

  $0.a_{11}a_{12}a_{13}a_{14}\ldots a_{1n}\ldots$

  $0.a_{21}a_{22}a_{23}a_{24}\ldots a_{2n}\ldots$

  $0.a_{31}a_{32}a_{33}a_{34}\ldots a_{3n}\ldots$

  … … … …

- Select the digits on the diagonal
- build a number

$$d_n = \begin{cases} 1 & if\ a_{nm} \neq 1 \\ 2 & if\ a_{nm} = 1 \end{cases}$$

- d differs in the n^{th} position from the n^{th} in the list

All Reals

- Card.($\{x \in R | 0 < x < 1\}$) = Card.(R)

Positive Rationals Q^+

- Card.(Q^+) =?= Card.(Z)

log function properties (from back cover of textbook)

definition of log: $\log_b x = y \leftrightarrow b^y = x$

$\log_b b = 1$	$\log_b x = \dfrac{\log_n x}{\log_n b}$
$\log_b x^a = a * \log_b x$	

$$\log_b (x * y) = \log_b x + \log_b y$$

$$\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$$

$$\log_b (x) = \log_b y \rightarrow x = y$$