Name (PRINTED): ______________________

Student ID #: ______________________

Section # (or TA's: name and time) ______________________

CMSC 250 | Exam #2 VERSION A | Thurs., Apr. 20, 2006

Write all answers legibly on the paper provided. If you need extra paper, raise your hand and request a blank paper – you must put your name on and hand-in any paper you receive. You can also use the back of the last page which is blank. Clearly label any answers that appear on a paper different from where the question appears. You must indicate the continuation of the answer on the paper where the question is and on the paper where the answer is continued. The number of points possible for each question is indicated in square brackets – the total number of points on the exam is 100, and you will have exactly 70 minutes to complete this exam. In order to receive any partial credit, you must show your work, clearly labeled in the space provided. You may not use calculators, textbooks or any other external aids during this exam. The formula sheet is attached - this can be removed from the back of the exam and does not need to be handed in at the end.

Important Note: The final exam will be on Tuesday, May 16 4:00-6:00 pm in CSIC 2117 for 0101 and 0102 (Jan Plane's sections) and CSIC 3117 for 0201 and 0202 (Clyde Kruskal's sections). Send mail to your instructor by May 1 if you have a conflict with another exam at that time.

**** This area is for grading purposes (points lost per page)- Do not write below this line ****

2	3	4	5	6	7	8	9	10	Total

1. [30 pnts.] Assume you have been given a friend's coin collection. The coin collection consissts of 70 coins: 2 silver dollars, 20 fifty-cent pieces, 10 quarters, 5 dimes, 8 nickels 25 pennies. Answer the following questions about the coins in that collection.

Answer the following questions by giving a solution taken to a form that includes no more than just addition, subtraction, multiplication, division, factorials and/or exponents. [note: two coins of the same value are considered to be identical and indistinguishable from each other unless the dates are specifically mentioned - then no two coins of the same value in the collection have the same date.] All questions are independent - do not assume the event in a previous question has happened when you are answering the next question. If you do not know exactly how to solve it, give an explanation in English of things you must consider for partial credit.

(a) His coin collection is held in one large jar. You reach in and pull one coin out at random. What is the probability that it is either a silver dollar or a fifty-cent piece? (remember: 2S, 20F, 10Q, 5D, 8N and 25P = 70 coins)

$$\frac{2+20}{70} = \frac{22}{70}$$

(b) The jar spills all over the floor, what is the probability that all of the coins land heads up? (remember: 2S, 20F, 10Q, 5D, 8N and 25P = 70 coins)

$$\left(\frac{1}{2}\right)^{70}$$

(c) Your friend has 7 great-grandchildren. You must distribute the pennies to those children in such a way that each child gets at least one. How many ways can you distribute them? (remember: 2S, 20F, 10Q, 5D, 8N and 25P = 70 coins)

$$25 - 7 = 18$$

$$\binom{18+7-1}{18} = \frac{(18+7-1)!}{18!(7-1)!} = \frac{24!}{18!6!}$$

(d) Your friend has 3 great-great-grandchildren. You must distribute the quarters and nickles to these 3 children. Since they are not old enough to realize what is happening, you do have the option of distributing them all to one child or any other combination. How many ways can you distribute those coins? (remember: 2S, 20F, 10Q, 5D, 8N and 25P = 70 coins)

$$\binom{10+3-1}{10}\binom{8+3-1}{8} = \frac{(10+3-1)!}{10!(3-1)!}\frac{8+3-1}{8!(3-1)!} = \frac{(12)!}{10!2!}\frac{10!}{8!2!} = \frac{12\cdot 11}{2}\frac{10\cdot 9}{2}$$

(e) You reach into the jar and pull out exactly six coins, what is the probability that you have exactly one of each in your hand? (remember: 2S, 20F, 10Q, 5D, 8N and 25P = 70 coins)

$$\frac{\binom{2}{1}\binom{20}{1}\binom{10}{1}\binom{5}{1}\binom{8}{1}\binom{25}{1}}{\binom{70}{6}} = \frac{2\cdot 20\cdot 10\cdot 5\cdot 8\cdot 25}{\frac{70!}{6!64!}}$$

(f) Assume 15 of the 25 pennies randomly drop into a single straight line on the floor. How many different ways could that line look assuming the dates on the coins make them all distinguishable? (remember: 2S, 20F, 10Q, 5D, 8N and 25P = 70 coins)

$$P(25,15) = \frac{25!}{(25-15)!} = \frac{25!}{10!}$$

(g) You put all 70 coins in a single file line. How many different ways can this line look? (note: two coins of the same value are indistinguishable from each other - you are not looking at the dates in this question.) (remember: 2S, 20F, 10Q, 5D, 8N and 25P = 70 coins)

$$\frac{70!}{2!20!10!5!8!25!}$$

(h) You pull two coins out of the jar at the same time, what is the probability that they have the same value? (remember: 2S, 20F, 10Q, 5D, 8N and 25P = 70 coins)

$$\frac{\binom{2}{2}+\binom{20}{2}+\binom{10}{2}+\binom{5}{2}+\binom{8}{2}+\binom{25}{2}}{\binom{70}{2}}$$
$$= \frac{2\cdot 1+20\cdot 19+10\cdot 9+5\cdot 4+8\cdot 7+25\cdot 24}{70\cdot 69}$$

$$\frac{2}{70}\frac{1}{69}+\frac{20}{70}\frac{19}{69}+\frac{10}{70}\frac{9}{69}+\frac{5}{70}\frac{4}{69}+\frac{8}{70}\frac{7}{69}+\frac{25}{70}\frac{24}{69}$$

(i) You ordered a pizza, and you need exactly twelve more cents in order to pay the delivery man (you must pay him exactly 12 cents). Assuming the date on the coin does matter for this question, how many different ways could you select coins to pay him? (remember: 2S, 20F, 10Q, 5D, 8N and 25P = 70 coins)

$$\binom{25}{12}+\binom{8}{1}\binom{25}{7}+\binom{8}{2}\binom{25}{2}+\binom{5}{1}\binom{25}{2} = \frac{25!}{12!13!}+\frac{8!}{1!7!}\frac{25!}{7!18!}+\frac{8!}{2!6!}\frac{25!}{2!23!}+\frac{5!}{1!4!}\frac{25!}{2!23!}$$

2. [20 pnts.] For each of the parts of this question: Either give a complete proof or a counter example. If you are using a counter example, you must give specific members for the sets A, B, C, D and U(the universal set) as needed or prove the following statements concerning Sets.

Be sure to give the name of the reason which justifies each step you give in the proof.

(a) $\forall A, B, C \in \{sets\},\ \ (A - B) \cup (B \cap C) = (C - (A \cup B)') \cup A$

FALSE.

Let $A = \{1\}$, $B = \{1\}$, and $C = \phi$.

$(A - B) \cup (B \cap C) \ = \ (\{1\} - \{1\}) \cup (\{1\} \cap \phi) \ = \ \phi \cup \phi \ = \ \phi$

$(C - (A \cup B)') \cup A \ = \ (\phi - (\{1\} \cup \{1\})') \cup \{1\} \ = \ (\phi - \{1\}') \cup \{1\} \ = \ \phi \cup \{1\} \ = \ \{1\}$

$\phi \neq \{1\}$ so this is a valid counter example.

(b) $\forall A, B, C, D \in \{sets\},\ \ (A - B) \cup (C - B) = ((A \cup B) \cup C) - B$

TRUE.

Proof.
Starting with the LHS:

$$\begin{aligned}
(A - B) \cup (C - B) &= (A \cap B') \cup (C \cap B') && \text{Alt rep of set diff (twice)} \\
&= (A \cup C) \cap B' && \text{Distributive} \\
&= ((A \cup C) \cap B') \cup \phi && \text{Union with empty set} \\
&= ((A \cup C) \cap B') \cup (B \cap B') && \text{Intersection with complement} \\
&= ((A \cup C) \cup B) \cap B' && \text{Distributive} \\
&= ((A \cup B) \cup C) \cap B' && \text{Associative and Commutative} \\
&= ((A \cup B) \cup C) - B && \text{Alt rep of set diff}
\end{aligned}$$

QED

Alternatively, starting with the RHS:

$$\begin{aligned}
((A \cup B) \cup C) - B &= ((A \cup B) \cup C) \cap B' && \text{Alt rep of set diff} \\
&= ((A \cup C) \cup B) \cap B' && \text{Associative and Commutative} \\
&= ((A \cup C) \cap B') \cup (B \cap B') && \text{Distributive} \\
&= ((A \cup C) \cap B') \cup \phi && \text{Intersection with complement} \\
&= (A \cup C) \cap B' && \text{Union with empty set} \\
&= (A \cap B') \cup (C \cap B') && \text{Distributive} \\
&= (A - B) \cup (C - B) && \text{Alt rep of set diff (twice)}
\end{aligned}$$

QED

3. [10 pnts] Given the sets: $A = \{a, b\}$, $B = \{1, 2\}$ and $C = \{\{1, 2\}, a, \emptyset\}$. Answer the following questions:

(a) Give the value of $A \times B$

$$\{(a, 1), (a, 2), (b, 1), (b, 2)\}$$

(b) Give the value of $P(A)$ [this is the powerset]

$$\{\phi, \{a\}, \{b\}, \{a, b\}\}$$

(c) Give the value of $P(C)$ [this is also the powerset]

$$\{\phi, \{\{1, 2\}\}, \{a\}, \{\phi\}, \{\{1, 2\}, a\}, \{\{1, 2\}, \phi\}, \{a, \phi\}, \{\{1, 2\}, a\}, \phi\}\}$$

(d) Answer with Yes or No: Is $B \in C$?

Yes.

(e) Answer with Yes or No: Is $B \subseteq C$?

No.

(f) Answer with Yes or No: Is $B \in P(C)$?

No.

(g) Answer with Yes or No: Is $B \subseteq P(C)$?

No.

4. [43 pnts.] For all of the parts of this question: Either find a specific counter example or prove each of the indicated statements is true. When using induction to prove something true, you must only use strong induction if it is required by that problem – using strong induction to prove something that only required regular induction, will result in a loss of points. You must use induction of some form to prove any and all of the following statements that are true.

(a) Assume that we know the sequence A_0, A_1, ... is defined as

$$A_0 = 12$$

$$A_1 = 29$$

$$\forall n \in Z^{\geq 2}, A_n = 5(A_{n-1}) - 6(A_{n-2})$$

Prove that the following statement is true or give a specific counter example to show that it is false:

$$\forall k \in Z^{\geq 0}, \quad A_k = 5(3^k) + 7(2^k)$$

TRUE.

Proof (by Strong Induction).
Base Case ($k = 0$, $k = 1$):
$k = 0$: $A_k = A_0 = 12$, $5(3^k) + 7(2^k) = 5(3^0) + 7(2^0) = 5(1) + 7(1) = 5 + 7 = 12$
$k = 1$: $A_k = A_1 = 29$, $5(3^k) + 7(2^k) = 5(3^1) + 7(2^1) = 5(3) + 7(2) = 15 + 14 = 29$
Induction Hypothesis: ($\forall i \in Z, \ \ 0 \leq i \leq p$)
$A_i = 5(3^i) + 7(2^i)$

Inductive Step ($k = p + 1$):
Show $A_{p+1} = 5(3^{p+1}) + 7(2^{p+1})$
Proof:

$$\begin{aligned}
A_{p+1} &= 5A_p - 6A_{p-1} \quad \text{by definition} \\
&= 5(5(3^p) + 7(2^p)) - 6(5(3^{p-1}) + 7(2^{p-1})) \quad \text{by the IH} \\
&= 25(3^p) + 35(2^p) - 30(3^{p-1}) - 42(2^{p-1})) \\
&= 25(3)(3^{p-1}) + 35(2)(2^{p-1}) - 30(3^{p-1}) - 42(2^{p-1})) \\
&= 75(3^{p-1}) + 70(2^{p-1}) - 30(3^{p-1}) - 42(2^{p-1})) \\
&= 45(3^{p-1}) + 28(2^{p-1}) \\
&= 5(3^2)(3^{p-1}) + 7(2^2)(2^{p-1}) \\
&= 5(3^{p+1}) + 7(2^{p+1})
\end{aligned}$$

QED

(b)

$$\forall n \in Z^{>-1},\ \ 1 + 3n \le 4^n$$

TRUE.

Proof (by Mathematical Induction).

Base Case ($k = 0$):
LHS: $1 + 3n \ = \ 1 + 3 \cdot 0 \ = \ 1 + 0 \ = \ 1$
RHS: $4^n \ = \ 4^0 \ = \ 1$
$1 \le 1$ so LHS $\le$ RHS.

Inductive Hypothesis (for $i = p$)
$1 + 3i \le 4^i$

Inductive Step ($k = p + 1$):
Show $1 + 3(p + 1) \le 4^{p+1}$, or equivalently $4^{p+1} \ge 3p + 4$
Proof:

$$\begin{aligned} 4^{p+1} &= 4 \cdot 4^p \quad \text{by algebra} \\ &\ge 4 \cdot (1 + 3p) \quad \text{by the IH} \\ &= 12p + 4 \quad \text{by algebra} \\ &= (3p + 4) + 9p \quad \text{by algebra} \\ &\ge 3p + 4 \quad \text{since } p \ge 0 \end{aligned}$$

QED

(c)

$$\forall n \in Z^{\geq 2}, \ \sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n-2)+6}{3}$$

FALSE.

Let $n = 3$.

LHS:

$$\sum_{i=1}^{n-1} i(i+1) \ = \ \sum_{i=1}^{3-1} i(i+1) \ = \ \sum_{i=1}^{2} i(i+1) \ = \ 1(1+1)+2(2+1) \ = \ 1(2)+2(3) \ = \ 2+6 \ = \ 8$$

RHS:

$$\frac{n(n-1)(n-2)+6}{3} \ = \ \frac{3(3-1)(3-2)+6}{3} \ = \ \frac{3(2)(1)+6}{3} \ = \ \frac{6+6}{3} \ = \ \frac{12}{3} \ = \ 4$$

Since $8 \neq 4$ this is a valid counter example.