CMSC 250 Homework 5 ANSWERS Spring 2006

Due Wed March 1 at the beginning of your discussion section.

For each of the following, state either that the statement is true or that the statement is false. Then give a complete, convincing proof to justify your answer.

1. For all integers x, if x is an even integer then $x^2 + x$ is an even integer.

   TRUE.
   This is the same as $\forall x \in Z, x \in Z^{even} \rightarrow x^2 + x \in Z^{even}$.

   Proof.

   Let x be arbitrary in the domain of integers.
   |Assmue x is even.
   |Then $x = 2y$ for some integer y by the definition of even.
   |Then $x^2 + x \; = \; (2y)^2 + (2y)$ by substitution.
   | $= \; 4y^2 + 2y$ by multiplication.
   | $= \; 2(2y^2 + y)$ by distribution.
   |Since $2y^2 + y \in Z$ by the closure of the integers in addition and multiplication,
   |$2(2y^2 + y) \in Z^{even}$ by the definition of even
   |and $x^2 + x \in Z^{even}$ by substitution of equalities.

   $x \in Z^{even} \rightarrow x^2 + x \in Z^{even}$ by closing the conditional world without a contradiction.
   Since x was arbitrary in the domain of integers, we can use generalizing from the generic particular to get:
   $\forall x \in Z, x \in Z^{even} \rightarrow x^2 + x \in Z^{even}$
   QED

2. The sum of any two rational numbers is rational.

   TRUE.
   This is the same as $\forall x, y \in Q, x + y \in Q$.

   Proof.

   Let x and y be arbitrary rational numbers.
   Then $\exists a, b, c, d \in Z$ such that $x = \frac{a}{b}$ and $y = \frac{c}{d}$
   where $b \neq 0$ and $d \neq 0$.
   Then $x + y \ = \ \frac{a}{b} + \frac{c}{d}$ by substitution
   $= \frac{ad}{bd} + \frac{bc}{bd}$ by multiplication of fractions
   $= \frac{ad+bc}{bd}$ by addition of fractions.

   Since the integers are closed under addition and multiplication, $\exists j \in Z, j = ad + bc$.
   Since the integers are closed under multiplication, $\exists k \in Z, k = bd$.
   Since neither b nor d are 0, $bd \neq 0$ by multiplication rules of algebra.

   Since $x + y \ = \ \frac{j}{k}$ where j and k are both integers and $k \neq 0$,
   $x + y \in Q$ by definition of rational.

   Since x and y were both arbitrary in Q, we can generalize to:
   $\forall x, y \in Q, x + y \in Q$.
   QED

3. The product of any two irrational numbers is irrational.

   FALSE.

   Disproof by counter example.
   Select $\sqrt{2}$.
   $\sqrt{2}$ is irrational,
   but $\sqrt{2} * \sqrt{2} = 2$, which is rational.

4. $\forall x, y, z \in Z, ((x|y) \wedge (x|z)) \rightarrow (x|y+z)$

   TRUE.

   Proof.

   Let x, y and z be arbitrary integers.
   $|$Assume $x|y$ and $x|z$
   $|\exists a, b \in Z$ such that $ax = y$ and $bx = z$ by definition of divides.
   $|y + z \ = \ ax + bx$ by substitution.
   $| \ = \ (a+b)x$ by distribution.
   $|$ Since $a + b \in Z$ by closure of Z in addition,
   $|x|y + z$, by definition of divides.

   $(x|y \wedge x|z) \rightarrow x|(y+z)$ by closing the conditional world without a contradiction.

   Since x, y and z were arbitrary integers, we can generalize from the generic particular and get:
   $\forall x, y, z \in Z, (x|y \wedge x|z) \rightarrow x|(y+z)$.

5. The sum of any two consecutive integers is odd.

   TRUE.

   Proof.

   Let n be an arbitrary integer.
   Then its successor is $n + 1$.
   The sum of these two consecutive integers is $n + (n + 1)$
   This equals $2n + 1$ by algebra.
   Since n was defined above as an integer, $2n + 1 \in Z^{odd}$by the definition of odd.
   Since n was arbitrary in the set we can generalize to:
   For all pairs of two consecutive integers, their sum is odd.

6. For all integers x greater than 3, $4|x(6 * 2^x)$

   TRUE.

   Proof.

   Let x be an arbitrary integer greater than 3.
   Then $x = y + 2$ for some integer $y > 1$.
   Therefore $x(6 * 2^x) \ = \ (y + 2)(6 * 2^{y+2})$ by substitution.
   $= \ (y + 2)(6 * 2^2 * 2^y)$ by algebra.
   $= \ 2^2(y + 2)(6 * 2^y)$ by associativity and commutativity.
   $= \ 4(y + 2)(6 * 2^y)$ by squaring the 2.

   Since $(y + 2)(6 * 2^y) \in Z$ by closure of Z in addition and multiplication,
   we know that $4|4(y + 2)(6 * 2^y)$ by definition of divides
   and $4|x(6 * 2^x)$ by substitution.
   QED

7. $\forall n \in Z, n \in Z^{even} \to n + n^2 + n^3 \in Z^{even}$

TRUE.

Proof:

Let n be arbitrary in Z.
$|$Assume $n \in Z^{even}$. $|\exists k \in Z, n = 2k$ by definition of even.
$|n + n^2 + n^3 \;=\; (2k) + (2k)^2 + (2k)^3$ by substitution.
$| \;=\; 2k + 4k^2 + 8k^3$ by algebra
$| \;=\; 2(k + 2k^2 + 4k^3)$ by distribution
$|$ Since $k + 2k^2 + 4k^3 \in Z$ by closure of Z in multiplication and addition,
$|2(k + 2k^2 + 4k^3) \in Z^{even}$ by the definition of even.
$|$ and $n + n^2 + n^3 \in Z^{even}$ by substitution.

Therefore $n \in Z^{even} \to n + n^2 + n^3 \in Z^{even}$ by closing the conditional world w/out contradiction.

Since n was arbitrary in the integers, we can generalize to:
$\forall n \in Z, n \in Z^{even} \to n + n^2 + n^3 \in Z^e ven$
QED

8. $\forall x, y \in Z^{odd}, x - y \in Z^{even}$

TRUE.

Proof.

Assume x and y are arbitrary in Z^{odd}.
Then $x = 2a + 1$, where $a \in Z$ by the definition of odd,
and $y = 2b + 1$, where $b \in Z$ by the definition of odd.
Then $x - y \;=\; (2a + 1) - (2b + 1)$ by substitution
$\;=\; 2a - 2b$ by algebra
$\;=\; 2(a - b)$ by distribution.
Since $a - b \in Z$ by closure of Z in subtraction,
$2(a - b) \in Z^{even}$ by the definition of even
and $x - y \in Z^{even}$ by substitution.
QED

9. $\forall n \in Z, n^2 - n > 0 \to n \neq 1$

TRUE.

We construct the contrapositive. We must show that

$$\forall n \in Z, \sim n \neq 1 \to \sim (n^2 - n > 0)$$

which is equivlaent to

$$\forall n \in Z, n = 1 \to n^2 - n \leq 0$$

Proof.

Let n be arbitrary in Z.

| Assume $n = 1$. |$n^2 - n = 1^2 - 1 = 1 - 1 = 0$ by algebra.

|$0 \leq 0$ by definition of less than or equal to.

|$n^2 - n \leq 0$ by substitution.

$n = 1 \rightarrow n^2 - n \leq 0$ by closing the conditional world without contradiction.
Since n was arbitrary in Z, we can generalize to:
$\forall n \in Z, n = 1 \rightarrow n^2 - n \leq 0$.
QED

10. If n is the sum of any four consecutive integers, $\exists k \in Z, n = 4k + 2$.

TRUE.

Proof.

Let m be arbitrary in the domain of integers.
Four consecutive integers can be written as m, $m + 1$, $m + 2$, and $m + 3$.
Then their sum is $m + (m + 1) + (m + 2) + (m + 3)$

$= 4m + 6 = 4m + 4 + 2 = 4(m + 1) + 2$ by algebra.

Select $k = m + 1$. $k \in Z$ by closure of Z in addition.
$m + (m + 1) + (m + 2) + (m + 3) = 4k + 2$ by substitution.
Since m was arbitrary in Z, this can be generalized to:
$\forall m \in Z, \exists k \in Z, m + (m + 1) + (m + 2) + (m + 3) = 4k + 2$.
QED

11. For all integers n, n is an even number if and only if $5n^2 + 12$ is even.

TRUE.

We must show that if n is even then $5n^2 + 12$ is even, and if $5n^2 + 12$ is even then n is even. For the second statement, we construct the contrapositive and show that if n is odd then $5n^2 + 12$ is odd. So we need to prove the following two statements:

$\forall n \in Z, n \in Z^{even} \rightarrow 5n^2 + 12 \in Z^{even}$

$\forall n \in Z, n \in Z^{odd} \rightarrow 5n^2 + 12 \in Z^{odd}$

Proof.

Let n be arbitrary in the domain of integers.

|Assume n is even.

|Then $n = 2m$ for some integer m.

|Then $5n^2 + 12 = 5(2m)^2 + 12$ by substitution

| $= 5 * 2^2 m^2 + 12 = 20m^2 + 12 = 2(10m^2 + 6)$ by algebra

|$10m^2 + 6 \in Z$ by the closure of the integers in addition and multiplication.

So $2(10m^2 + 6) \in Z^{even}$ by the definition of even

and $5n^2 + 12 \in Z^{even}$ by substitution of equalities.

$n \in Z^{even} \rightarrow 5n^2 + 12 \in Z^{even}$ by closing the conditional world without a contradiction.

Since n was arbitrary in the domain of integers, we can use generalizing from the generic particular to get:

$\forall n \in Z, n \in Z^{even} \rightarrow 5n^2 + 12 \in Z^{even}$

Let n be arbitrary in the domain of integers.

|Assume n is odd.

|Then $n = 2m + 1$ for some integer m.

|Then $5n^2 + 12 = 5(2m + 1)^2 + 12$ by substitution

| $= 5 * (4m^2 + 4m + 1) + 12 = 20m^2 + 20m + 16 + 1 = 2(10m^2 + 10m + 8) + 1$ by algebra

|$10m^2 + 10m + 8 \in Z$ by the closure of the integers in addition and multiplication.

So $2(10m^2 + 10m + 8) + 1 \in Z^{odd}$ by the definition of odd

and $5n^2 + 12 \in Z^{odd}$ by substitution of equalities.

$n \in Z^{odd} \rightarrow 5n^2 + 12 \in Z^{odd}$ by closing the conditional world without a contradiction.

Since n was arbitrary in the domain of integers, we can use generalizing from the generic particular to get:

$\forall n \in Z, n \in Z^{odd} \rightarrow 5n^2 + 12 \in Z^{odd}$

QED

12. $\forall m \in Z, \exists k \in Z, m^2 = 5k \vee m^2 = 5k + 1 \vee m^2 = 5k + 4$.

TRUE.

Proof.

Let m be arbitrary in the domain of integers.
$m = 5a \ \vee \ m = 5a + 1 \ \vee m = 5a + 2 \ \vee \ m = 5a + 3 \ \vee \ m = 5a + 4$

Case $m = 5a$:
$m^2 \ = \ (5a)^2$ by substitution
$\quad = \ 25a^2 \ = \ 5(5a^2)$ by algebra
Let $k = 5a^2$
$k \in Z$ by the closure of the integers in multiplication.
$m^2 \ = \ 5k$ by substitution
$m^2 = 5k \vee m^2 = 5k + 1 \vee m^2 = 5k + 4$ by disjunctive addition.
$\exists k \in Z, m^2 = 5k \vee m^2 = 5k + 1 \vee m^2 = 5k + 4$ by existential generalization.

Case $m = 5a + 1$:
$m^2 \ = \ (5a + 1)^2$ by substitution
$\quad = \ 25a^2 + 10a + 1 \ = \ 5(5a^2 + 2a) + 1$ by algebra
Let $k = 5a^2 + 2a$
$k \in Z$ by the closure of the integers in addition and multiplication.
$m^2 \ = \ 5k + 1$ by substitution
$m^2 = 5k \vee m^2 = 5k + 1 \vee m^2 = 5k + 4$ by disjunctive addition.
$\exists k \in Z, m^2 = 5k \vee m^2 = 5k + 1 \vee m^2 = 5k + 4$ by existential generalization.

Case $m = 5a + 2$:
$m^2 \ = \ (5a + 2)^2$ by substitution
$\quad = \ 25a^2 + 20a + 4 \ = \ 5(5a^2 + 4a) + 4$ by algebra
Let $k = 5a^2 + 4a$
$k \in Z$ by the closure of the integers in addition and multiplication.
$m^2 \ = \ 5k + 4$ by substitution
$m^2 = 5k \vee m^2 = 5k + 1 \vee m^2 = 5k + 4$ by disjunctive addition.
$\exists k \in Z, m^2 = 5k \vee m^2 = 5k + 1 \vee m^2 = 5k + 4$ by existential generalization.

Case $m = 5a + 3$:
$m^2 \ = \ (5a + 3)^2$ by substitution
$\quad = \ 25a^2 + 30a + 9 \ = \ 5(5a^2 + 6a + 1) + 4$ by algebra
Let $k = 5a^2 + 6a + 1$
$k \in Z$ by the closure of the integers in addition and multiplication.
$m^2 \ = \ 5k + 1$ by substitution
$m^2 = 5k \vee m^2 = 5k + 1 \vee m^2 = 5k + 4$ by disjunctive addition.
$\exists k \in Z, m^2 = 5k \vee m^2 = 5k + 1 \vee m^2 = 5k + 4$ by existential generalization.

Case $m = 5a + 4$:
$m^2 \ = \ (5a + 4)^2$ by substitution
$\quad = \ 25a^2 + 40a + 16 \ = \ 5(5a^2 + 8a + 3) + 1$ by algebra
Let $k = 5a^2 + 8a + 3$
$k \in Z$ by the closure of the integers in addition and multiplication.
$m^2 \ = \ 5k + 1$ by substitution
$m^2 = 5k \vee m^2 = 5k + 1 \vee m^2 = 5k + 4$ by disjunctive addition.
$\exists k \in Z, m^2 = 5k \vee m^2 = 5k + 1 \vee m^2 = 5k + 4$ by existential generalization.

Therefore, $\exists k \in Z, m^2 = 5k \vee m^2 = 5k + 1 \vee m^2 = 5k + 4$ by proof by divistion into cases.
Since m was arbitrary in the domain of integers, we can use generalizing from the generic particular to get:
$\forall m \in Z, \exists k \in Z, m^2 = 5k \vee m^2 = 5k + 1 \vee m^2 = 5k + 4.$