# CMSC 250 Homework 8 Spring 2003

## Due Wed March 29 at the beginning of your discussion section.

**You must write the solutions to the problems single-sided on your own lined paper, with all sheets stapled together, and with all answers written in sequential order or you will lose points.**

For each of the following statements say whether it is TRUE or FALSE. If true, use *mathematical induction* to prove it. If false, give a specific counter example.

1. $\forall n \in Z^{\geq 1},\ \sum_{i=1}^{n}(2i+1) = (n+1)^2 - 1.$

   TRUE.

   Proof (by mathematical induction)

   **Base Case:** (n=1)
   LHS: $\sum_{i=1}^{n}(2i+1) \ = \ 2*1+1 \ = \ 3$
   RHS: $(n+1)^2 - 1 \ = \ (1+1)^2 - 1 \ = \ 3$
   LHS = RHS since $3 \ = \ 3$.

   **Inductive Hypothesis:**
   Assume true for $n-1$. $\sum_{i=1}^{n-1}(2i+1) \ = \ ((n-1)+1)^2 - 1 \ = \ n^2 - 1$.

   **Inductive Step:**

$$\begin{aligned}
\sum_{i=1}^{n}(2i+1) &= \sum_{i=1}^{n-1}(2i+1) \ + \ 2n+1 && \text{by splitting summation} \\
&= n^2 - 1 \ + \ 2n+1 && \text{by the IH} \\
&= n^2 - 1 \ + \ 2n+1 && \text{by algebra} \\
&= n^2 + 2n + 1 - 1 && \text{by algebra} \\
&= (n+1)^2 - 1 \ + \ 2n+1 && \text{by algebra}
\end{aligned}$$

   QED

2. $\forall n \in Z^{\geq 1}, 7 | 2^{3n} - 1.$

   TRUE.

   Proof (by mathematical induction)

   **Base Case:** (n=1)
   $2^{3n} - 1 \ = \ 2^{3*1} - 1 \ = \ 7.$
   $7|7.$

**Inductive Hypothesis:**
Assume true for $n-1$. Then $7|2^{3(n-1)}-1$.

**Inductive Step:**
Since $7|2^{3(n-1)}-1$, there exists $k \in Z$ such that $2^{3(n-1)}-1 \ = \ 7k$.

$$\begin{aligned}
2^{3n}-1 &= 2^3 * 2^{3(n-1)} - 1 && \text{by algebra} \\
&= 2^3 * 2^{3(n-1)} - 2^3 + 2^3 - 1 && \text{by algebra} \\
&= 2^3(2^{3(n-1)}-1) + 2^3 - 1 && \text{by algebra} \\
&= 2^3(2^{3(n-1)}-1) + 7 && \text{by algebra} \\
&= 2^3(7k) + 7 && \text{by substitution}
\end{aligned}$$

QED

3. $\forall n \in Z^+, \ \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

   TRUE.

   **Base Case:** (n=1)
   LHS: $\sum_{i=1}^{n} \frac{1}{i(i+1)} \ = \ \sum_{i=1}^{1} \frac{1}{i(i+1)} \ = \ \frac{1}{1(1+1)} \ = \ \frac{1}{2)}$.
   RHS: $\frac{n}{n+1} \ = \ \frac{1}{1+1} \ = \ \frac{1}{2}$
   LHS = RHS since $\frac{1}{2} \ = \ \frac{1}{2}$.

   **Inductive Hypothesis:**
   Assume true for $n$. Then $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.

   **Inductive Step:**

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)((n+1)+1)} && \text{by splitting summation} \\
&= \frac{n}{n+1} + \frac{1}{(n+1)((n+1)+1)} && \text{by IH} \\
&= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} && \text{by algebra} \\
&= \frac{n(n+2) \ + \ 1}{(n+1)(n+2)} && \text{by algebra} \\
&= \frac{n^2+2n+1}{(n+1)(n+2)} && \text{by algebra} \\
&= \frac{(n+1)^2}{(n+1)(n+2)} && \text{by algebra} \\
&= \frac{n+1}{n+2} && \text{by algebra} \\
&= \frac{n+1}{(n+1)+1} && \text{by algebra}
\end{aligned}$$

   QED

4. $\forall n \in Z^{\geq 2}, 3|4^n + 1$.

   FALSE.

Proof (by counterexample)
$n = 2:$
$4^n + 1 \ = \ 4^2 + 1 \ = \ 17.$
$3 \not| 17.$
QED

5. $\forall n \in Z^{\geq 3}, \ \ n^2 \geq 2n + 3$

   TRUE.

   Proof (by mathematical induction)

   **Base Case:** (n=3)
   LHS: $n^2 \ = \ 3^2 \ = \ 9$
   RHS: $2n + 3 \ = \ 2 * 3 + 3 \ = \ 9$
   LHS $\geq$ RHS since $9 \ \geq \ 9$.

   **Inductive Hypothesis:**
   Assume true for $n - 1$. Then $(n-1)^2 \geq 2(n-1) + 3$.

   **Inductive Step:**

$$
\begin{aligned}
n^2 &= n^2 - 2n + 1 + 2n - 1 && \text{by algebra} \\
&= (n-1)^2 + 2n - 1 && \text{by algebra} \\
&\geq 2(n-1) + 3 + 2n - 1 && \text{by IH} \\
&= 4n && \text{by algebra} \\
&= 2n + 3 + 2n - 3 && \text{by algebra} \\
&\geq 2n + 3 && \text{since } 2n > 3 \text{ for } n > 3
\end{aligned}
$$

   QED

6. $\forall n \in Z^{\geq 0}, \ \ 2(n+2) \leq (n+2)^2$

   TRUE.

   Proof (by mathematical induction)

   **Base Case:** (n=0)
   LHS: $2(n+2) \ = \ 2(0+2) \ = \ 4$
   RHS: $(n+2)^2 \ = \ (0+2)^2 \ = \ 4$
   LHS $\leq$ RHS since $4 \ \leq \ 4$.

   **Inductive Hypothesis:**
   Assume true for $n - 1$. Then $2((n-1)+2) \leq ((n-1)+2)^2$ or $2(n+1) \leq ((n+1)^2$.

   **Inductive Step:**
   We reverse the inequality.

$$
\begin{aligned}
(n+2)^2 &= n^2 + 4n + 4 && \text{by algebra} \\
&= n^2 + 2n + 1 + 2n + 3 && \text{by algebra}
\end{aligned}
$$

$$\begin{aligned}
&= (n+1)^2 + 2n + 3 && \text{by algebra}\\
&\geq 2(n+1) + 2n + 3 && \text{by IH}\\
&= 4n + 4 && \text{by algebra}\\
&= 2n + 4 + 2n && \text{by algebra}\\
&= 2(n+2) + 2n && \text{by algebra}\\
&\geq 2(n+2) && \text{since } n > 0
\end{aligned}$$

QED

7. $\forall n \in Z^{\geq 2},\ \sum_{k=2}^{n} \frac{1}{k-1} = \frac{n}{2}$

   FALSE.

   Let $n = 4$.
   LHS: $\sum_{k=2}^{n} \frac{1}{k-1} = \sum_{k=2}^{4} \frac{1}{k-1} = 1 + 1/2 + 1/3 = 11/6$.
   RHS: $\frac{4}{2} = 2$.
   LHS $\neq$ RHS since $11/6 \neq 2$.

8. $\forall n \in Z^{\geq 0},\ 2|(n+2)!$

   TRUE.

   Proof (by mathematical induction)

   **Base Case:** (n=0)
   $(n+2)! = (0+2)! = 2! = 2$.
   $2|2$.

   **Inductive Hypothesis:**
   Assume true for $n-1$. Then $2|((n-1)+2)!$ or $2|(n+1)!$.

   **Inductive Step:**
   Since $2|(n+1)!$ (by IH), there exists $k \in Z$ such that $(n+1)! = 2k$ (by definition of divides).
   Thus $(n+1)! = 2k$.
   Thus $(n+2)(n+1)! = (n+2)2k$.
   Thus $(n+2)! = 2(n+2)k$.
   Since $Z$ is closed under addition and multiplication, $(n+2)k$ is an integer.
   So $2|(n+2)!$ by definition of divides.
   QED