# CMSC 250 Homework 10 Spring 2006

## Due Wed April 12 at the beginning of your discussion section.

**You must write the solutions to the problems single-sided on your own lined paper, with all sheets stapled together, and with all answers written in sequential order or you will lose points.**

1. Prove each statement is true or find a specific counterexample. Assume all sets are subsets of a universal set U.

   (a) $\forall A, B \in \{sets\},\ A - B \neq B - A$
   FALSE.

   Let $A = B = \emptyset$.
   Then $A - B = \emptyset - \emptyset = \emptyset$, and $B - A = \emptyset - \emptyset = \emptyset$.
   But $\emptyset = \emptyset$.

   (b) $\forall A, B, C \in \{sets\},\ (A \cup (B \cap C)) \subseteq (A \cup B \cup C)$
   TRUE.
   Proof.

   Let $A, B, C$ be arbitrary $\in \{sets\}$.
   Let $x$ be an arbitrary element of the universe $U$.
   | Assume $x \in A \cup (B \cap C)$.
   | $x \in A \ \vee\ x \in (B \cap C)$ by Definition of Union.
   |
   | Case $x \in A$:
   | $x \in A \vee x \in B$ by Disjunctive Addition.
   | $x \in A \vee x \in B \vee x \in C$ by Disjunctive Addition.
   | $x \in A \vee x \in (B \cup C)$ by Definition of Union.
   | $x \in A \cup (B \cup C)$ by Definition of Union.
   |
   | Case $x \in (B \cap C)$:
   | $x \in B \wedge x \in C$ by Definition of Intersection.
   | $x \in B$ by Conjunctive Simplification.
   | $x \in B \vee x \in C$ by Disjunctive Addition.
   | $x \in A \vee (x \in B \vee x \in C)$ by Disjunctive Addition.
   | $x \in A \vee x \in (B \cup C)$ by Definition of Union.
   | $x \in A \cup (B \cup C)$ by Definition of Union.
   |
   | Therefore, $x \in A \cup (B \cup C)$ by Proof by Division into Cases.
   | $x \in A \cup B \cup C$ by Associative Law.
   $x \in A \cup (B \cap C) \ \rightarrow\ (x \in A \cup B \cup C)$ by CCW without a Contradiction.
   $\forall x \in U,\ x \in A \cup (B \cap C) \ \rightarrow\ (x \in A \cup B \cup C)$ by Universal Generalization.
   $(A \cup (B \cap C)) \subseteq (A \cup B \cup C)$ by Definition of Subset.
   $\forall A, B, C \in \{sets\},\ (A \cup (B \cap C)) \subseteq (A \cup B \cup C)$ by Universal Generalization.
   QED

Alternative proof (of same theorem).

Let $A, B, C$ be arbitrary $\in \{sets\}$.
Let $x$ be an arbitrary element of the universe $U$.
| Assume $x \in A \cup (B \cap C)$.
| $x \in ((A \cup B) \cap (A \cup C))$ by Distributive Law.
| $x \in (A \cup B) \wedge x \in (A \cup C)$ by Definition of Intersection.
| $x \in (A \cup B)$ by Conjunctive Simplification.
| $x \in (A \cup B) \vee x \in C$ by Disjunctive Addition.
| $x \in (A \cup B) \cup C$ by Definition of Union.
| $x \in A \cup B \cup C$ by Associative Law.
$x \in A \cup (B \cap C) \ \rightarrow \ (x \in A \cup B \cup C)$ by CCW without a Contradiction.
$\forall x \in U, \ x \in A \cup (B \cap C) \ \rightarrow \ (x \in A \cup B \cup C)$ by Universal Generalization.
$(A \cup (B \cap C)) \subseteq (A \cup B \cup C)$ by Definition of Subset.
$\forall A, B, C \in \{sets\}, \ (A \cup (B \cap C)) \subseteq (A \cup B \cup C)$ by Universal Generalization.
QED

(c) $\forall A, B, C \in \{sets\}, \ (A - B) - C \subseteq (A - C)$

TRUE.

Proof.

Let $A, B, C$ be arbitrary $\in \{sets\}$.
Let $x$ be an arbitrary element of the universe $U$.
| Assume $x \in (A - B) - C$.
| $x \in (A - B) \cap C'$ by Definition of Set Difference.
| $x \in (A \cap B') \cap C'$ by Definition of Set Difference.
| $x \in (A \cap B') \wedge x \in C'$ by Definition of Intersection.
| $x \in (A \cap B')$ by Conjunctive Simplification.
| $x \in A \wedge x \in B'$ by Definition of Intersection.
| $x \in A$ by Conjunctive Simplification.
| $x \in C'$ by Conjunctive Simplification.
| $x \in A \wedge x \in C'$ by Conjunctive Addition.
| $x \in A \cap C'$ by Definition of Intersection.
| $x \in A - C$ by Definition of Set Difference.
$x \in (A - B) - C \rightarrow x \in A - C$ by CCW without a Contradiction.
$\forall x \in U, \ x \in (A - B) - C \rightarrow x \in A - C$ by Universal Generalization.
$(A - B) - C \subseteq A - C$ by Definition of Subset.
$\forall A, B, C \in \{sets\}, \ (A - B) - C \subseteq A - C$ by Universal Generalization.
QED

(d) $\forall A, B, C \in \{sets\}, \ (A \times B) \times C = A \times (B \times C)$

FALSE.

Let $U = \{u\}$. Then

$$(A \times B) \times C \ = \ \{(u, u), u\}$$

and

$$A \times (B \times C) \ = \ \{u, (u, u)\}.$$

The two sets each have one element, but $((u, u), u) \neq (u, (u, u))$.

(e) $\forall A, B, C \in \{sets\},\ A - (B \cup (A \cap C)) = (A - (B \cup C)) \cup (A \cap B \cap C)$
FALSE.

Let $A = B = \{x\}$.
Then the LHS

$$\begin{aligned} A - (B \cup (A \cap C)) &= \{x\} - (\{x\} \cup (\{x\} \cap \{x\})) \\ &= \{x\} - (\{x\} \cup (\{x\}) \\ &= \{x\} - \{x\} \\ &= \emptyset \end{aligned}$$

And the RHS

$$\begin{aligned} (A - (B \cup C)) \cup (A \cap B \cap C) &= (\{x\} - (\{x\} \cup \{x\})) \cup (\{x\} \cap \{x\} \cap \{x\}) \\ &= (\{x\} - \{x\}) \cup \{x\} \\ &= \emptyset \cup \{x\} \\ &= \{x\} \end{aligned}$$

LHS $\neq$ RHS.

2. Find each of the following assuming $A = \{\{1\}, a\}$ and $B = \{A, 1\}$

   (a) Find $\mathcal{P}(A)$
   $\mathcal{P}(A) \ = \ \{\emptyset, \{\{1\}\}, \{a\}, \{\{1\}, a\}\}$

   (b) Find $\mathcal{P}(B)$
   $\mathcal{P}(B) \ = \ \{\emptyset, \{A\}, \{1\}, \{A, 1\}\}$

   (c) Answer the following questions:
      i. Is $A \in \mathcal{P}(B)$? No.
      ii. Is $A \subseteq \mathcal{P}(B)$? No.
      iii. Is $A \subseteq B$? No.
      iv. Is $A \in B$? Yes.

3. Let $A = \{a, b, c, d, e, f\}$, $A_1 = \{a, c\}$, $A_2 = \{b, d\}$, $A_3 = \{e, f\}$, $A_4 = \{a, c, d\}$.

   (a) Explain why $\{A_1, A_2, A_3, A_4\}$ does not form a partition of $A$.
   The elements $a, c$ are in both $A_1$ and $A_4$. Also, the element $d$ is in both $A_2$ and $A_4$.

   (b) Explain why $\{A_3, A_4\}$ does not form a partition of $A$.
   The element $b$ is not in any set.

   (c) Explain why $\{A_1, A_2, A_3\}$ does form a partition of $A$.
   Every element occurs in some set exactly once.

   (d) Alter the set $A_3$ in two different ways so that the new collection of subsets $\{A_1, A_2, A_3\}$ fails to be a partition of $A$ for a different reason for each of the alterations.
   Remove $e$ (from $A_3$): Then $e$ does not occur in any set.
   Add $a$ (into $A_3$): Then $a$ occurs in both $A_1$ and $A_3$.

4. Your UMCP student ID is an 8 digit number.

   (a) How many different ID's are possible (assuming no other restrictions)?
   $10^8$

   (b) How many different ID's are possible if there can be no 0's in the number?
   $9^8$

   (c) How many different ID's are possible if the number can neither start nor end with a 0?
   $9^2 10^6$

   (d) What is the probability that a ID chosen at random contains no 0's?
   $\frac{9^8}{10^8} = (9/10)^8$