CMSC 330: Organization of Programming Languages

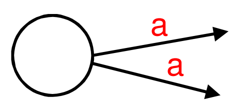
Finite Automata 2

Types of Finite Automata

- Deterministic Finite Automata (DFA)
  - Exactly one sequence of steps for each string
  - All examples so far
- Nondeterministic Finite Automata (NFA)
  - May have many sequences of steps for each string
  - Accepts if any path ends in final state at end of string
  - More compact than DFA

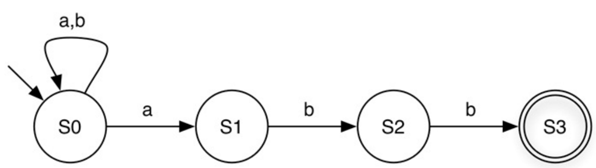
Comparing DFAs and NFAs

- NFAs can have more than one transition leaving a state on the same symbol



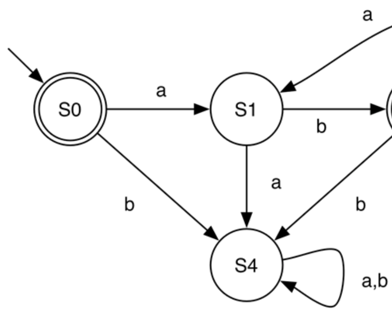
- DFAs allow only one transition per symbol
  - I.e., transition function must be a valid function
  - DFA is a special case of NFA

NFA for (a|b)*abb



- ba
  - Has paths to either S0 or S1
  - Neither is final, so rejected
- babaabb
  - Has paths to different states
  - One path leads to S3, so accepts string

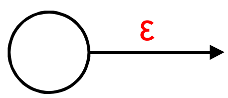
Another example DFA



- Language?
  - (ab|aba)*

Comparing DFAs and NFAs (cont.)

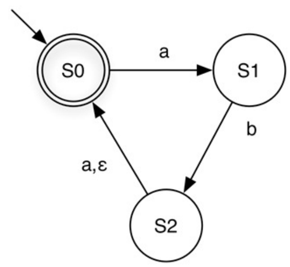
- NFAs may have transitions with empty string label
  - May move to new state without consuming character



ε-transition

- DFA transition must be labeled with symbol
  - DFA is a special case of NFA

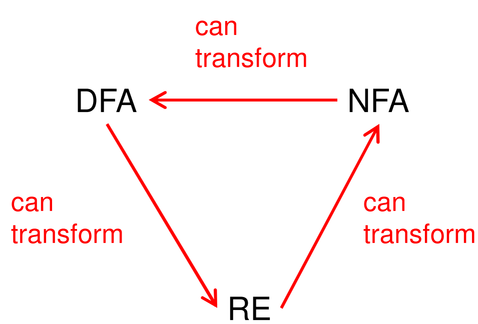
NFA for (ab|aba)*



- aba
  - Has paths to states S0, S1
- ababa
  - Has paths to S0, S1
  - Need to use ε-transition

Relating REs to DFAs and NFAs

- Regular expressions, NFAs, and DFAs accept the same languages!



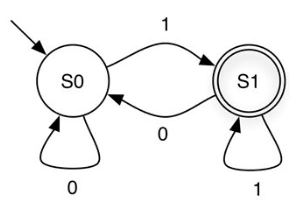
Formal Definition

- A deterministic finite automaton *(DFA)* is a 5-tuple $(\Sigma, Q, q_0, F, \delta)$ where
  - Σ is an alphabet
    - the strings recognized by the DFA are over this set
  - Q is a nonempty set of states
  - $q_0 \in Q$ is the start state
  - $F \subseteq Q$ is the set of final states
    - How many can there be?
  - $\delta : Q \times \Sigma \to Q$ specifies the DFA's transitions
    - What's this definition saying that δ is?
- A DFA accepts s if it stops at a final state on s

Formal Definition: Example

- $\Sigma = \{0, 1\}$
- $Q = \{S0, S1\}$
- $q_0 = S0$
- $F = \{S1\}$
-

δ	0	1
S0	S0	S1
S1	S0	S1

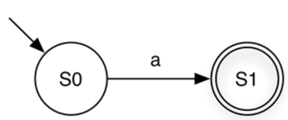


Nondeterministic Finite Automata (NFA)

- An *NFA* is a 5-tuple $(\Sigma, Q, q_0, F, \delta)$ where
  - Σ is an alphabet
  - Q is a nonempty set of states
  - $q_0 \in Q$ is the start state
  - $F \subseteq Q$ is the set of final states
  - $\delta \subseteq Q \times (\Sigma \cup \{\varepsilon\}) \times Q$ specifies the NFA's transitions
    - Transitions on ε are allowed – can optionally take these transitions without consuming any input
    - Can have more than one transition for a given state and symbol
- An NFA accepts s if there is at least one path from its start to final state on s

Reducing Regular Expressions to NFAs

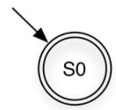
- Goal: Given regular expression e, construct NFA: <e> = $(\Sigma, Q, q_0, F, \delta)$
  - Remember regular expressions are defined recursively from primitive RE languages
  - Invariant: $|F| = 1$ in our NFAs
    - Recall F = set of final states
- Base case: a



<a> = ({a}, {S0, S1}, S0, {S1}, {(S0, a, S1)})

Reduction (cont.)

- Base case: ε



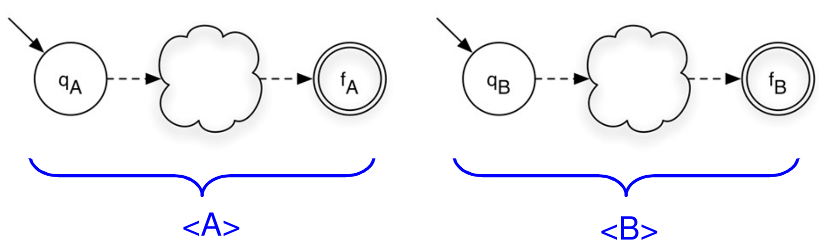
<ε> = (ε, {S0}, S0, {S0}, ∅)

- Base case: ∅

<∅> = (∅, {S0, S1}, S0, {S1}, ∅)

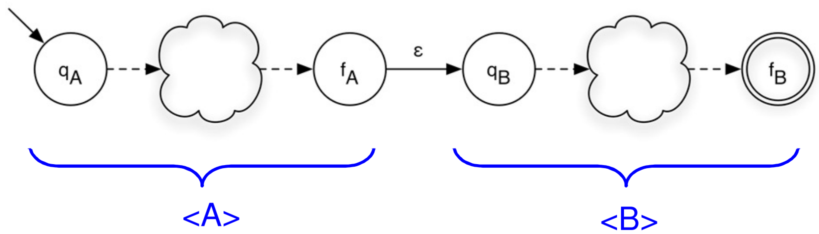
Reduction: Concatenation

- Induction: AB



Reduction: Concatenation (cont.)

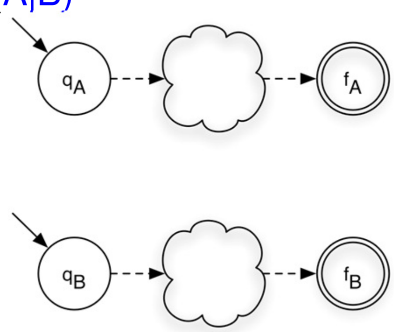
- Induction: AB



- <A> = $(\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$
- <B> = $(\Sigma_B, Q_B, q_B, \{f_B\}, \delta_B)$
- <AB> = $(\Sigma_A \cup \Sigma_B, Q_A \cup Q_B, q_A, \{f_B\}, \delta_A \cup \delta_B \cup \{(f_A,\varepsilon,q_B)\})$

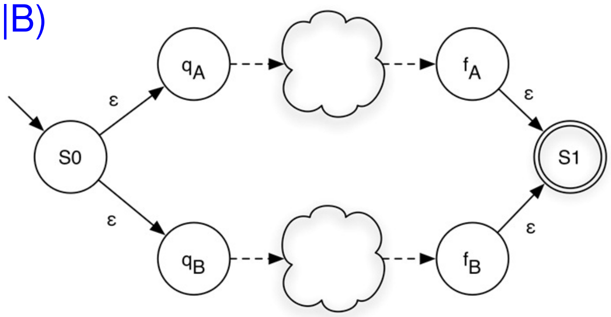
Reduction: Union

- Induction: (A|B)



Reduction: Union (cont.)

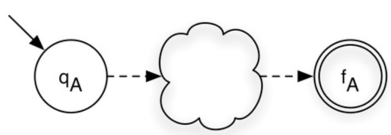
- Induction: (A|B)



  - <A> = $(\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$
  - <B> = $(\Sigma_B, Q_B, q_B, \{f_B\}, \delta_B)$
  - <(A|B)> = $(\Sigma_A \cup \Sigma_B, Q_A \cup Q_B \cup \{S0,S1\}, S0, \{S1\},$
    $\delta_A \cup \delta_B \cup \{(S0,\varepsilon,q_A), (S0,\varepsilon,q_B), (f_A,\varepsilon,S1), (f_B,\varepsilon,S1)\})$

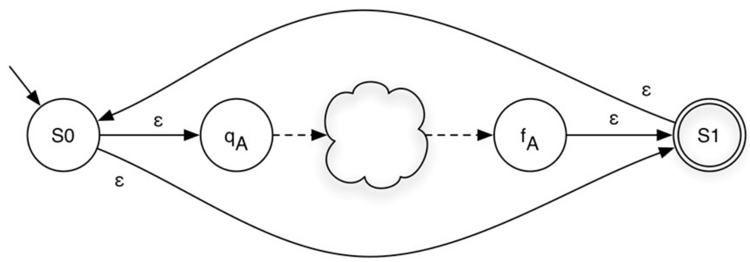
Reduction: Closure

- Induction: A*



Reduction: Closure (cont.)

- Induction: A*



- $\langle A \rangle = (\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$
- $\langle A^* \rangle = (\Sigma_A, Q_A \cup \{S0,S1\}, S0, \{S1\},$
  $\delta_A \cup \{(f_A,\varepsilon,S1), (S0,\varepsilon,q_A), (S0,\varepsilon,S1), (S1,\varepsilon,S0)\})$

Reduction Complexity

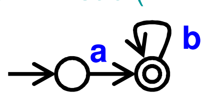
- Given a regular expression A of size n...
  Size = # of symbols + # of operations
- How many states does <A> have?
  - 2 added for each |, 2 added for each *
  - O(n)
  - That's pretty good!

Practice

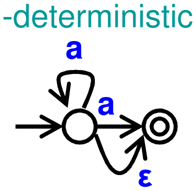
- Draw NFAs for the following regular expressions and languages
  - (0|1)*110*
  - 101*|111
  - all binary strings ending in 1 (odd numbers)
  - all alphabetic strings which come after “hello” in alphabetic order
  - (ab*c|d*a|ab)d

Recap

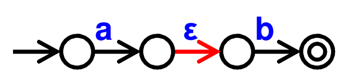
- Finite automata
  - Alphabet, states…
  - (Σ, Q, q_0, F, δ)
- Types
  - Deterministic (DFA)



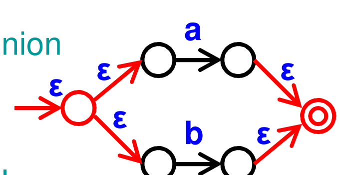
  - Non-deterministic (NFA)



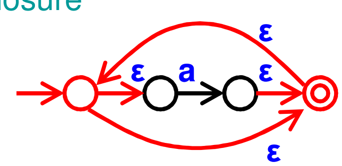
- Reducing RE to NFA
  - Concatenation



  - Union



  - Closure



Next

- Reducing NFA to DFA
  - ε-closure
  - Subset algorithm
- Minimizing DFA
  - Hopcroft reduction
- Complementing DFA
- Implementing DFA

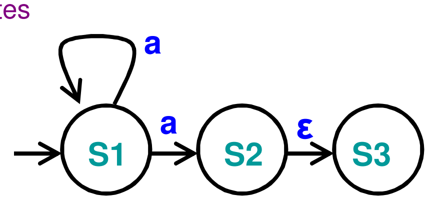
How NFA Works

- When NFA processes a string
  - NFA may be in several possible states
    - Multiple transitions with same label
    - ε-transitions
- Example
  - After processing “a”
    - NFA may be in states

      S1

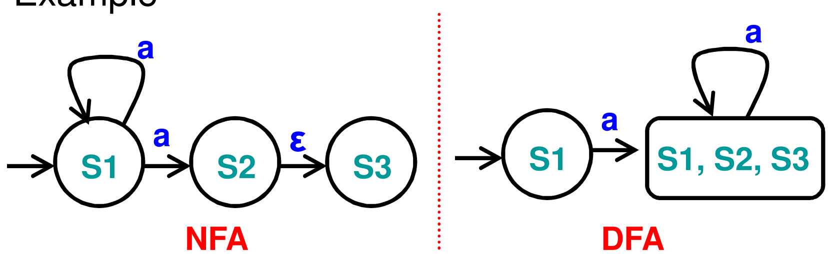
      S2

      S3



Reducing NFA to DFA

- NFA may be reduced to DFA
  - By explicitly tracking the set of NFA states
- Intuition
  - Build DFA where
    - Each DFA state represents a set of NFA states
- Example



Reducing NFA to DFA (cont.)

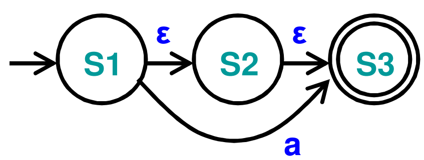
- Reduction applied using the subset algorithm
  - DFA state is a subset of set of all NFA states
- Algorithm
  - Input
    - NFA (Σ, Q, q_0, F_n, δ)
  - Output
    - DFA (Σ, R, r_0, F_d, δ)
  - Using
    - ε-closure(p)
    - move(p, a)

ε-transitions and ε-closure

- We say $p \xrightarrow{\varepsilon} q$
  - If it is possible to go from state p to state q by taking only ε-transitions
  - If $\exists$ $p, p_1, p_2, \dots p_n, q \in Q$ such that
    - $\{p,\varepsilon,p_1\} \in \delta$, $\{p_1,\varepsilon,p_2\} \in \delta$, … , $\{p_n,\varepsilon,q\} \in \delta$
- ε-closure(p)
  - Set of states reachable from p using ε-transitions alone
    - Set of states q such that $p \xrightarrow{\varepsilon} q$
    - ε-closure(p) = $\{q \mid p \xrightarrow{\varepsilon} q\}$
  - Note
    - ε-closure(p) always includes p
    - ε-closure( ) may be applied to set of states (take union)

ε-closure: Example 1

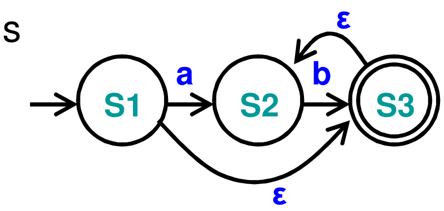
- Following NFA contains
  - $S1 \xrightarrow{\varepsilon} S2$
  - $S2 \xrightarrow{\varepsilon} S3$
  - $S1 \xrightarrow{\varepsilon} S3$



- ε-closures
  - ε-closure(S1) = { S1, S2, S3 }
  - ε-closure(S2) = { S2, S3 }
  - ε-closure(S3) = { S3 }
  - ε-closure({ S1, S2 }) = { S1, S2, S3 } ∪ { S2, S3 }

ε-closure: Example 2

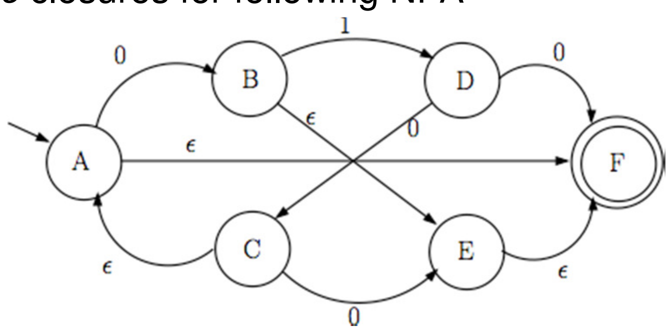
- Following NFA contains
  - $S1 \xrightarrow{\varepsilon} S3$
  - $S3 \xrightarrow{\varepsilon} S2$
  - $S1 \xrightarrow{\varepsilon} S2$



- ε-closures
  - ε-closure(S1) = { S1, S2, S3 }
  - ε-closure(S2) = { S2 }
  - ε-closure(S3) = { S2, S3 }
  - ε-closure({ S2,S3 }) = { S2 } ∪ { S2, S3 }

ε-closure: Practice

- Find ε-closures for following NFA



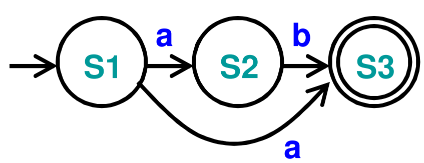
- Find ε-closures for the NFA you construct for
  - The regular expression (0|1*)111(0*|1)

Calculating move(p,a)

- move(p,a)
  - Set of states reachable from p using exactly one transition on a
    - Set of states q such that $\{p, a, q\} \in \delta$
    - $\text{move}(p,a) = \{q \mid \{p, a, q\} \in \delta\}$
  - Note move(p,a) may be empty Ø
    - If no transition from p with label a

move(a,p) : Example 1

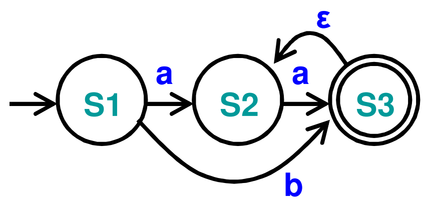
- Following NFA
  - $\Sigma = \{ a, b \}$



- Move
  - move(S1, a) = { S2, S3 }
  - move(S1, b) = Ø
  - move(S2, a) = Ø
  - move(S2, b) = { S3 }
  - move(S3, a) = Ø
  - move(S3, b) = Ø

move(a,p) : Example 2

- Following NFA
  - $\Sigma = \{ a, b \}$



- Move
  - move(S1, a) = { S2 }
  - move(S1, b) = { S3 }
  - move(S2, a) = { S3 }
  - move(S2, b) = Ø
  - move(S3, a) = Ø
  - move(S3, b) = Ø

NFA → DFA Reduction Algorithm

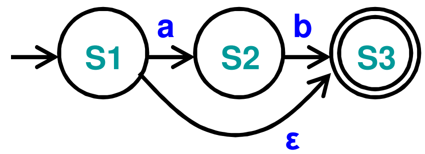
- Input NFA $(\Sigma, Q, q_0, F_n, \delta)$, Output DFA $(\Sigma, R, r_0, F_d, \delta)$
- Algorithm

Let $r_0 = \varepsilon$-closure(q_0), add it to R	// DFA start state
While $\exists$ an unmarked state $r \in R$	// process DFA state r
Mark r	// each state visited once
For each $a \in \Sigma$	// for each letter a
Let $S = \{s \mid q \in r \ \& \ move(q,a) = s\}$	// states reached via a
Let $e = \varepsilon$-closure(S)	// states reached via ε
If $e \notin R$	// if state e is new
Let $R = e \cup R$	// add e to R (unmarked)
Let $\delta = \delta \cup \{r, a, e\}$	// add transition r→e
Let $F_d = \{r \mid \exists s \in r \text{ with } s \in F_n\}$	// final if include state in F_n

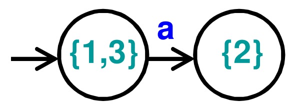
NFA → DFA Example 1

- Start = ε-closure(S1) = { {S1,S3} }
- R = { {S1,S3} }
- r ∈ R = {S1,S3}
- Move({S1,S3},a} = {S2}
  - e = ε-closure({S2}) = {S2}
  - R = R ∪ {S2} = { {S1,S3}, {S2} }
  - δ = δ ∪ {{S1,S3}, a, {S2}}
- Move({S1,S3},b} = Ø

NFA



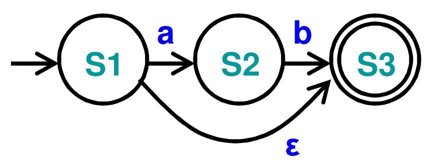
DFA



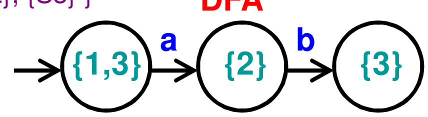
NFA → DFA Example 1 (cont.)

- R = { {S1,S3}, {S2} }
- r ∈ R = {S2}
- Move({S2},a} = Ø
- Move({S2},b} = {S3}
  - e = ε-closure({S3}) = {S3}
  - R = R ∪ {S3} = { {S1,S3}, {S2}, {S3} }
  - δ = δ ∪ {{S2}, b, {S3}}

NFA



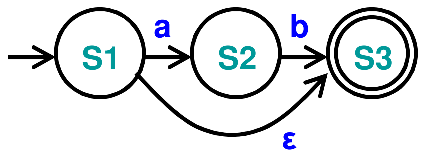
DFA



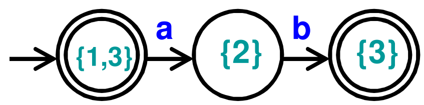
NFA → DFA Example 1 (cont.)

- R = { {S1,S3}, {S2}, {S3} }
- r ∈ R = {S3}
- Move({S3},a) = Ø
- Move({S3},b) = Ø
- F_d = {{S1,S3}, {S3}}
  - Since S3 ∈ F_n
- Done!

NFA

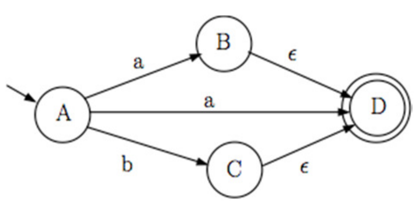


DFA

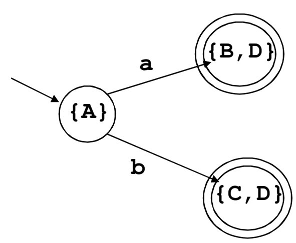


NFA → DFA Example 2

- NFA



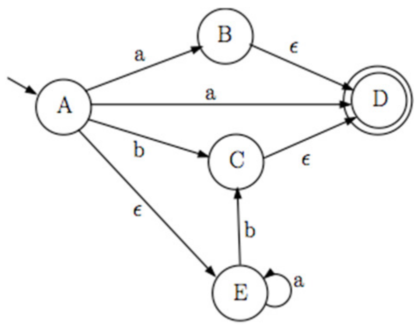
- DFA



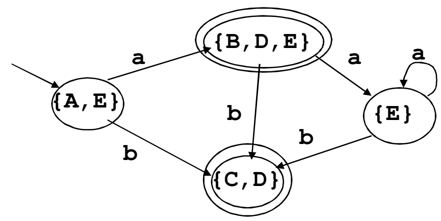
R = { {A}, {B,D}, {C,D} }

NFA → DFA Example 3

- NFA



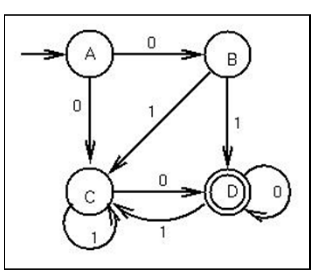
- DFA



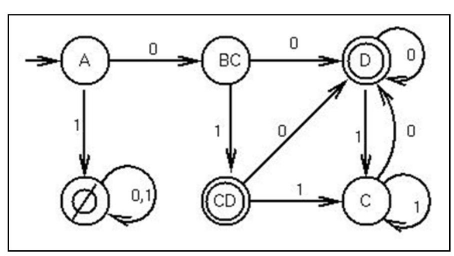
R = { {A,E}, {B,D,E}, {C,D}, {E} }

Equivalence of DFAs and NFAs

- Any string from {A} to either {D} or {CD}
  - Represents a path from A to D in the original NFA



NFA



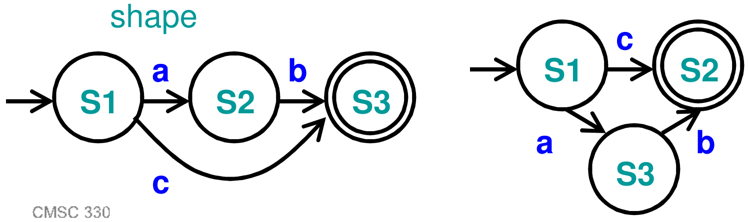
DFA

Equivalence of DFAs and NFAs (cont.)

- Can reduce any NFA to a DFA using subset alg.
- How many states in the DFA?
  - Each DFA state is a subset of the set of NFA states
  - Given NFA with n states, DFA may have 2^n states
    - Since a set with n items may have 2^n subsets
  - Corollary
    - Reducing a NFA with n states may be $O(2^n)$

Minimizing DFA

- Result from CS theory
  - Every regular language is recognizable by a minimum-state DFA that is unique up to state names
- In other words
  - For every DFA, there is a unique DFA with minimum number of states that accepts the same language
  - Two minimum-state DFAs have same underlying shape



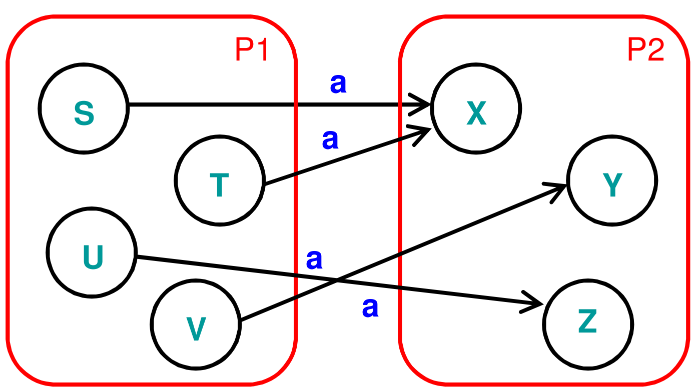
J. Hopcroft, "An n log n algorithm for minimizing states in a finite automaton," 1971

Minimizing DFA: Hopcroft Reduction

- Intuition
  - Look for states that can be distinguish from each other
    - End up in different accept / non-accept state with identical input
- Algorithm
  - Construct initial partition
    - Accepting & non-accepting states
  - Iteratively refine partitions (until partitions remain fixed)
    - Split a partition if members in partition have transitions to different partitions for same input
      - Two states x, y belong in same partition if and only if for all symbols in Σ they transition to the same partition
  - Update transitions & remove dead states

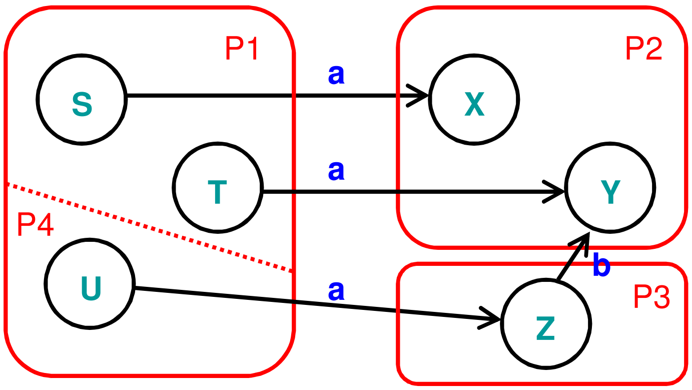
Splitting Partitions

- No need to split partition {S,T,U,V}
  - All transitions on a lead to identical partition P2
  - Even though transitions on a lead to different states



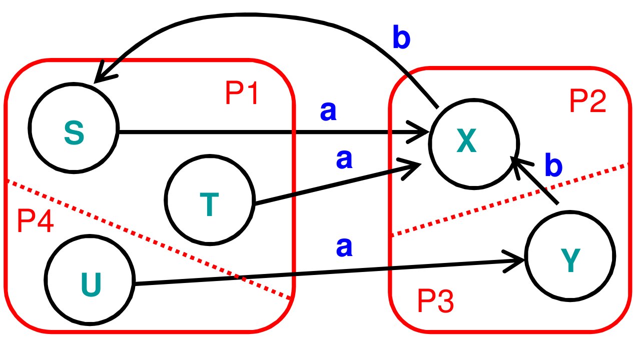
Splitting Partitions (cont.)

- Need to split partition {S,T,U} into {S,T}, {U}
  - Transitions on a from S,T lead to partition P2
  - Transition on a from U lead to partition P3



Resplitting Partitions

- Need to reexamine partitions after splits
  - Initially no need to split partition {S,T,U}
  - After splitting partition {X,Y} into {X}, {Y}
  - Need to split partition {S,T,U} into {S,T}, {U}



DFA Minimization Algorithm (1)

- Input DFA (Σ, Q, q_0, F_n, δ), Output DFA (Σ, R, r_0, F_d, δ)
- Algorithm

Let $p_0 = F_n$, $p_1 = Q - F$ // initial partitions = final, nonfinal states
Let $R = \{ p \mid p \in \{p_0,p_1\}$ and $p \mathrel{!}= \varnothing \}$, $P = \varnothing$ // add p to R if nonempty
While P != R do // while partitions changed on prev iteration
  Let $P = R$, $R = \varnothing$ // P = prev partitions, R = current partitions
  For each $p \in P$ // for each partition from previous iteration
    $\{p_0,p_1\}$ = split(p,P) // split partition, if necessary
    $R = R \cup \{ p \mid p \in \{p_0,p_1\}$ and $p \mathrel{!}= \varnothing \}$ // add p to R if nonempty
$r_0 = p \in R$ where $q_0 \in p$ // partition w/ starting state
$F_d = \{ p \mid p \in R$ and exists $s \in p$ such that $s \in F_n \}$ // partitions w/ final states
$\delta(p,c) = q$ when $\delta(s,c) = r$ where $s \in p$ and $r \in q$ // add transitions

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DFA Minimization Algorithm (2)

- Algorithm for split(p,P)

Choose some $r \in p$, let $q = p - \{r\}$, $m = \{ \}$ // pick some state r in p
For each $s \in q$ // for each state in p except for r
  For each $c \in \Sigma$ // for each symbol in alphabet
    If $\delta(r,c) = q_0$ and $\delta(s,c) = q_1$ and // q's = states reached for c
      there is no $p_1 \in P$ such that $q_0 \in p_1$ and $q_1 \in p_1$ then
      $m = m \cup \{s\}$ // add s to m if q's not in same partition
Return $p - m$, m // m = states that behave differently than r
// m may be $\varnothing$ if all states behave the same
// p – m = states that behave the same as r

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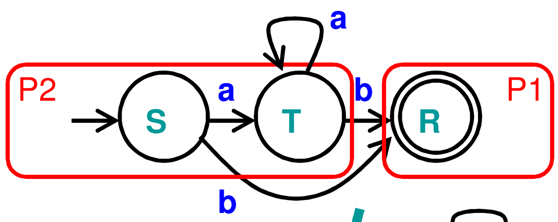
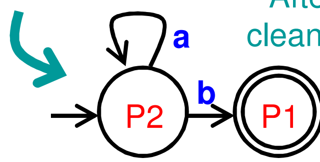
Minimizing DFA: Example 1

- DFA



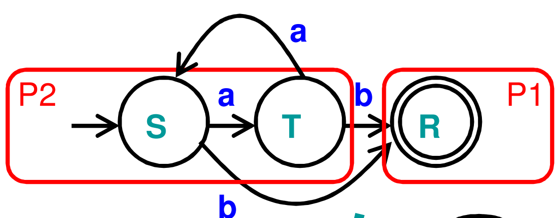
After cleanup



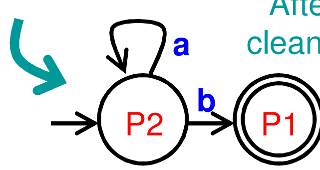
- Initial partitions
  - Accept { R } → P1
  - Reject { S, T } → P2
- Split partition? → Not required, minimization done
  - move(S,a) = T → P2 – move(S,b) = R → P1
  - move(T,a) = T → P2 – move (T,b) = R → P1

Minimizing DFA: Example 2

- DFA



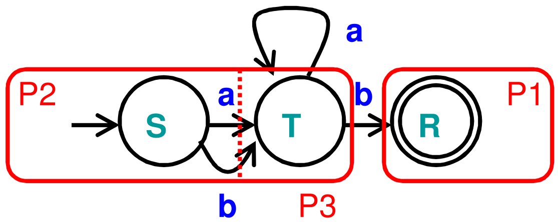
After cleanup



- Initial partitions
  - Accept { R } → P1
  - Reject { S, T } → P2
- Split partition? → Not required, minimization done
  - move(S,a) = T → P2 – move(S,b) = R → P1
  - move(T,a) = S → P2 – move (T,b) = R → P1

Minimizing DFA: Example 3

- DFA

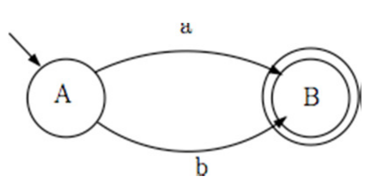


DFA already minimal

- Initial partitions
  - Accept { R } → P1
  - Reject { S, T } → P2
- Split partition? → Yes, different partitions for B
  - move(S,a) = T → P2
    - move(S,b) = T → P2
  - move(T,a) = T → P2
    - move (T,b) = R → P1

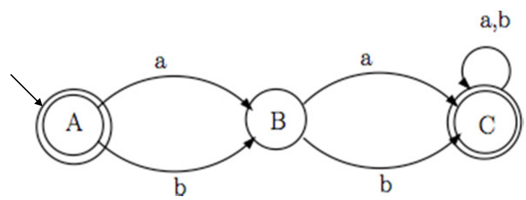
Complement of DFA

- Given a DFA accepting language L
  - How can we create a DFA accepting its complement?
  - Example DFA
    - $\Sigma = \{a,b\}$



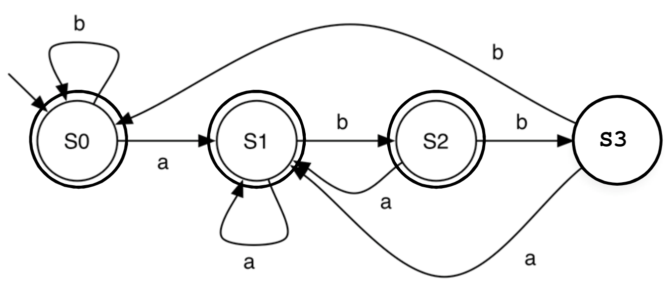
Complement of DFA (cont.)

- Algorithm
  - Add explicit transitions to a dead state
  - Change every accepting state to a non-accepting state & every non-accepting state to an accepting state
- Note this only works with DFAs
  - Why not with NFAs?



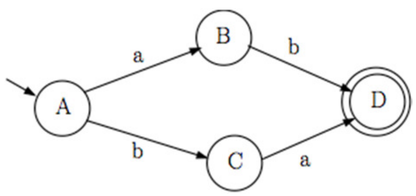
Practice

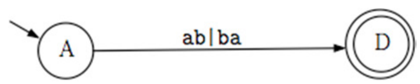
Make the DFA which accepts the complement of the language accepted by the DFA below.



Reducing DFAs to REs

- General idea
  - Remove states one by one, labeling transitions with regular expressions
  - When two states are left (start and final), the transition label is the regular expression for the DFA



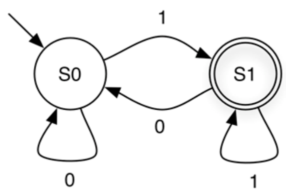


Relating REs to DFAs and NFAs

- Why do we want to convert between these?
  - Can make it easier to express ideas
  - Can be easier to implement

Implementing DFAs

It's easy to build a program which mimics a DFA




```
cur_state = 0;
while (1) {

  symbol = getchar();

  switch (cur_state) {

    case 0: switch (symbol) {
              case '0':  cur_state = 0; break;
              case '1':  cur_state = 1; break;
              case '\n': printf("rejected\n"); return 0;
              default:   printf("rejected\n"); return 0;
            }
            break;

    case 1: switch (symbol) {
              case '0':  cur_state = 0; break;
              case '1':  cur_state = 1; break;
              case '\n': printf("accepted\n"); return 1;
              default:   printf("rejected\n"); return 0;
            }
            break;

    default: printf("unknown state; I'm confused\n");
             break;
  }
}
```


Implementing DFAs (Alternative)

Alternatively, use generic table-driven DFA

given components (Σ, Q, q_0, F, δ) of a DFA:
let q = q_0
while (there exists another symbol s of the input string)
    q := δ(q, s);
if q $\in$ F then
    accept
else reject

- q is just an integer
- Represent δ using arrays or hash tables
- Represent F as a set

Run Time of DFA

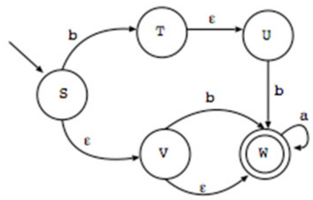
- How long for DFA to decide to accept/reject string s?
  - Assume we can compute $\delta(q, c)$ in constant time
  - Then time to process s is $O(|s|)$
    - Can't get much faster!
- Constructing DFA for RE A may take $O(2^{|A|})$ time
  - But usually not the case in practice
- So there's the initial overhead
  - But then processing strings is fast

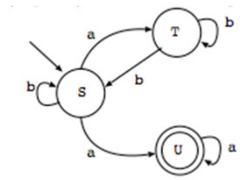
Regular Expressions in Practice

- Regular expressions are typically "compiled" into tables for the generic algorithm
  - Can think of this as a simple byte code interpreter
  - But really just a representation of $(\Sigma, Q_A, q_A, \{f_A\}, \delta_A)$, the components of the DFA produced from the RE
- Regular expression implementations often have extra constructs that are non-regular
  - I.e., can accept more than the regular languages
  - Can be useful in certain cases
  - Disadvantages
    - Nonstandard, plus can have higher complexity

Practice

- Convert to a DFA





- Convert to an NFA and then to a DFA
  - (0|1)*11|0*
  - Strings of alternating 0 and 1
  - aba*|(ba|b)

Summary of Regular Expression Theory

- Finite automata
  - DFA, NFA
- Equivalence of RE, NFA, DFA
  - RE → NFA
    - Concatenation, union, closure
  - NFA → DFA
    - ε-closure & subset algorithm
- DFA
  - Minimization, complement
  - Implementation