Announcements

- Homework #2 has been posted

Writing Proofs

A good proof should have:

- a statement of what is to be proven
- "Proof:" to indicate where the proof starts
- a clear indication of flow
- a clear indication of the reason for each step
- careful notation, completeness and order
- a clear indication of the conclusion

Statement of Claims

The following are equivalent:

- Every even number (greater than 3) is the sum of two primes.
- For all $n \in \mathbb{N}^{Even}$: If $n>3$, then n is the sum of two primes.
- $(\forall n \in \mathbb{N}^{Even})[n>3 \rightarrow (\exists a,b \in \mathbb{N}^{Prime})[n=a+b]]$
- $(\forall n \in \mathbb{N})[(\exists k \in \mathbb{N})[n=2k] \wedge n>3 \rightarrow (\exists a,b \in \mathbb{N})[a>1 \wedge (\forall c,d \in \mathbb{N})[cd=a \rightarrow c=1 \vee d=1] \wedge b>1 \wedge (\forall e,f \in \mathbb{N})[ef=b \rightarrow e=1 \vee f=1] \wedge n=a+b]]$

Which of these would you use to state your claim?

Constructive Proofs of Existence

- Claim: $(\exists a,b \in \mathbb{N})[a^b = b^a \wedge a \neq b]$
- Claim: There exist three natural numbers, a, b, and c (all distinct) such that $a^2 + b^2 = c^2$
- Claim: 23 can be written as the sum of 9 cubes (of non-negative integers).
- Claim: There is a number that can be written as the sum of two cubes (of positive integers) *in two different ways.*
- Talk about “Taxicab” numbers, and “non-constructive” proofs of existence.

Proofs by Exhaustion/Cases

- Claim: $\forall n \in \{1, 2, 3, 4\}$ $[(n + 1)^3 \geq 3^n]$.
- Claim: There are no integer solutions to the equation $a^2 + b^2 = 7$
- Claim: 23 cannot be written as the sum of 8 cubes (of non-negative integers).
- Mention proof of four-color problem

Applying Universal Generalization

- The most common technique for proving *universally quantified* statements.
- If you're not sure how to start – try this!

Claim: $(\forall x \in D)[P(x)]$

Proof:

Let $d \in D$, **arbitrarily chosen.**

…

$P(d)$

Since d was chosen arbitrarily, $P(x)$ holds for all $x \in D$.

Example of Proving a Universal Statement

Claim: $(\forall n \in \mathbb{N}^{Even})[n^2$ is even$])$

Proof Example: Rigid Style

Claim: **$(\forall n \in \mathbb{N}^{Even})[n^2$ is even$])$**

Proof:

(1) Let $a \in \mathbb{N}^{Even}$, selected arbitrarily
(2) $a = 2k$, for some $k \in \mathbb{N}$ [Defn of "even"]
(3) $a^2 = (2k)(2k)$
(4) $a^2 = 2(2k^2)$
(5) (Note that $2k^2 \in \mathbb{N}$) [Since $\mathbb{N}$ is closed under multiplication]
(6) a^2 is even [Defn of "even"; using (4), (5)]
(7) Since a was chosen arbitrarily, $(\forall n \in \mathbb{N}^{Even})[n^2$ is even$])$

Proof Example: Flowing Style

Claim: **The square of any even natural number is even.**

Proof:

Let $a \in \mathbb{N}^{Even}$, selected arbitrarily. Since *a* is even, a = 2k for some $k \in \mathbb{N}$. Squaring both sides, we get $a^2 = (2k)^2 = 2(2k^2)$. Noting that $2k^2 \in \mathbb{N}$ ($\mathbb{N}$ is closed under multiplication), we see that a^2 is equal to twice a natural number, hence a^2 is even. Since *a* was selected arbitrarily, the proposition holds for any even number.

- Which style do you think is easier to understand?
- Which style is easier to write without making mistakes?
- Can we use a style that is somewhere between these two?

More Examples

- Claim: The product of two odd integers is odd.
- Claim: $\mathbb{Q}$ is closed under multiplication. (Assuming we know that $\mathbb{Z}$ is closed under multiplication.)
- Claim: $(\forall n \in \mathbb{N}^{>0})$, $n^2 + 3n + 2$ is composite.