Announcements

- Homework #3 has been posted

One More Basic Example

- Claim: $\mathbb{Q}$ is dense. (Assuming we already know about the closure of $\mathbb{Q}$.)

More with Cases

- Claim: For all integers, *x* and *y*, $|x/y| = |x|/|y|$
- Claim: For all $n \in \mathbb{N}$, $3n^2 + n + 14$ is even.

Fun Proof

An existence proof that is as close to “constructive” as you can get without actually being constructive…

- Claim: There are two irrational numbers, a and b, such that a^b is rational.

Notation for “divisibility”

Suppose ab=c, where a,b,c$\in\mathbb{Z}$ (with b≠0)

- We use the following notation to express “b divides c”

  $b|c$

- In proofs, we frequently use the following interchangeably:

  $b|c$ is the same as $(\exists a\in\mathbb{Z})[\ c = ab]$

Implications

Below are outlines of the standard technique for proving implications:

Claim: If P then Q.
Proof:
Assume P.
…
Q.
Therefore, $P \rightarrow Q$.

Claim: $(\forall x \in D)$ [If P(x) then Q(x)].
Proof:
Let $d \in D$, selected arbitrarily.
Assume P(d) holds.
…
Q(d).
Therefore, $P(d) \rightarrow Q(d)$.
Since d was selected arbitrarily,
$(\forall x \in D)\ [P(x) \rightarrow Q(x)]$.

Examples with implications

- Claim: $\forall x,y,z \in \mathbb{N}$: If $x|y$, and $y|z$, then $x|z$.
- Claim: $\forall x,y \in \mathbb{R}$, if $x + y = 7$ and $xy = 10$, then $x^2 + y^2 = 29$

Proof by Contrapositive

Sometimes implications are easier to prove this way:

<u>Claim:</u> If P then Q.

<u>Proof:</u>

Assume ~Q.

…

~P.

Therefore, $P \rightarrow Q$.

Examples using Contrapositive

- $(\forall n \in \mathbb{N})$ If $3n + 2$ is odd then n is odd.
- $(\forall n, a, b \in \mathbb{R}^{+})$ If $n = ab$ then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Proofs of Equivalence ("If and only if")

Two techniques:

Claim: $P \leftrightarrow Q$.

Proof:

$P \leftrightarrow S1$

$\leftrightarrow S2$

$\leftrightarrow S3$

…

$\leftrightarrow Q$

- Doesn't always work
- Easy to make mistakes
- Maybe less writing

Claim: $P \leftrightarrow Q$.

Proof:

Part I. [Show $P \rightarrow Q$]

…

Part II. [Show $Q \rightarrow P$]

- Works more often
- Less error prone
- Probably more writing

Be careful!

Critique this "proof".

Warning: This proof is invalid!

Claim: $(\forall n \in \mathbb{N})[n \text{ is odd} \leftrightarrow n^2 \text{ is odd}]$

Proof:

Let $a \in \mathbb{N}$, selected arbitrarily.

a is odd $\leftrightarrow a = 2k + 1$ (some $k \in \mathbb{N}$)

$\leftrightarrow a^2 = (2k + 1)(2k + 1)$

$\leftrightarrow a^2 = 4k^2 + 4K + 1$

$\leftrightarrow a^2 = 2(2k^2 + 2k) + 1$

$\leftrightarrow a^2$ is odd (since $2k^2+2k \in \mathbb{N}$, by closure)

Since a was selected arbitrarily, the proposition is true for all $n \in \mathbb{N}$.

Proofs of Equivalence (if and only if)

- Claim: $(\forall n \in \mathbb{N})$[n is odd $\leftrightarrow$ n^2 is odd]
- Claim: $(\forall n,m \in \mathbb{N})$[n and m have the same “parity” $\leftrightarrow$ n + m is even]