

Stable Marriage and Man-Optimality

Definition 1. A man m and a woman w are **valid partners** means there exists a stable matching in which they are paired with each other.

Definition 2. For every man m , m 's **best valid partner** (denoted $best(m)$) is the highest-ranked valid partner of m , with respect to m 's preference list.

Definition 3. A matching S is **man-optimal** means that each man m is paired with $best(m)$ in S .

We have already seen that the Gale-Shapley algorithm (GS) always returns a matching that is stable. We will need that to prove the following.

Claim 1. For any fixed rule dictating the order of proposals, the matching S^* returned by GS is man-optimal.

Proof. First note that since S^* is stable, everyone is paired with one of their valid partners. Suppose by way of contradiction that there exists a man who is not paired with his best valid partner. Then he must be paired with a valid partner who comes after his best valid partner in his preference list. In other words, if S^* is not man-optimal, then at least one man was rejected by his best valid partner during the execution of GS.

Consider the *first* time it happens that some man m is rejected by his best valid partner, $w = best(m)$: w rejects m to be (or to continue to be) with someone else m' whom she prefers to m . Let us call this episode Event X.

Since w and m are valid partners, there exists a stable matching S' in which w is paired with m . In S' , m' is paired with someone else, say $w' \neq w$. w' is a valid partner of m' since S' is stable.

Now consider what the execution of GS tells us about m' 's preference between w and w' . **Event X was the first time in the execution of GS that any man was rejected by his best valid partner. In particular, at the time that Event X occurred, both the following are true:**

- m' has not been rejected by his $best(m')$ and therefore has not been rejected by any of his valid partners, in particular w' ;
- m' is paired with w , i.e. m' was rejected by every woman before w in his preference list.

Therefore, the execution of GS tells us that w' must be after w in m' 's preference list, i.e. m' prefers w to w' .

However, this contradicts the stability of S' : $(m, w), (m', w') \in S'$, but both w and m' prefer each other to their respective partners in S' . Therefore, our initial assumption that some man is rejected by his best valid partner during the execution of GS is false, i.e. S^* is man-optimal.

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