CMSC451 Homework 1
Spring 2015 due February 5, 2015

For all algorithms, provide time complexity analysis as well as a formal proof of correctness.

1. Staple all homework pages together and put your full name on each one (5 pts).

2. Read one of the articles listed under HW1 on the class webpage. Specify the article you read and then in 1-2 sentences, give your impressions (5 pts).

3. Given a sequence of integers $x_1, x_2, \ldots, x_n$ (possibly including negative integers) and an interval $I = [i, j]$, write x_I to denote the sum $\sum_{i \leq k \leq j} x_k$. Give an $O(n)$ time algorithm to find the interval that maximizes x_I.

4. Consider the problem of assigning children to hospital beds. You have n children and $n/2$ hospital rooms, each room with two beds (assume n is even). Each child has a preference list on the other $n - 1$ children as potential roommates and you want to find a stable assignment of children to children so that no two children have incentive to ditch their roommates for each other. Prove or find a counterexample for the claim that on any instance, there always exists a stable matching.

5. Suppose you have n input wires and n output wires, each directed from a source to a terminus. Each input wire meets each output wire in exactly one distinct point, at a special piece of hardware called a *junction box*. Points on the wire are naturally ordered in the direction from source to terminus; for two distinct points x and y on the same wire, we say that x is *upstream* from y if x is closer to the source than y, and otherwise we say x is *downstream* from y. The order in which one input wire meets the output wires in not necessarily the same as the order in which another input wire meets the output wires. (And similarly for the orders in which output wires meet input wires.)

   Each input wire is carrying a distinct data stream, and this data stream must be *switched* onto one of the output wires. If the stream of Input i is switched onto Output j at junction box B, then this stream passes through all the junction boxes upstream of B on Input i and all the junction boxes downstream from B on Output j. It does not matter which input data stream gets switched onto which output wire, but each input data stream must be switched onto a *different* output wire. Furthermore, no two data streams can pass through the same junction box following the switching operation. See Figure 1.

   Show that for any specified pattern in which the input wires and output wires meet each other (each pair meeting exactly once), a valid switching of the data streams can always be found – one in which each input data stream is switched onto a different output, and no two of the resulting streams pass through the same junction box. Additionally, give an algorithm to find such a valid switching and prove its correctness.

6. Let G be a graph on n nodes, where n is an even number. Prove or provide a counterexample for the following claim: if every node of G has degree at least $n/2$, then G is connected. (G is not a multigraph.)

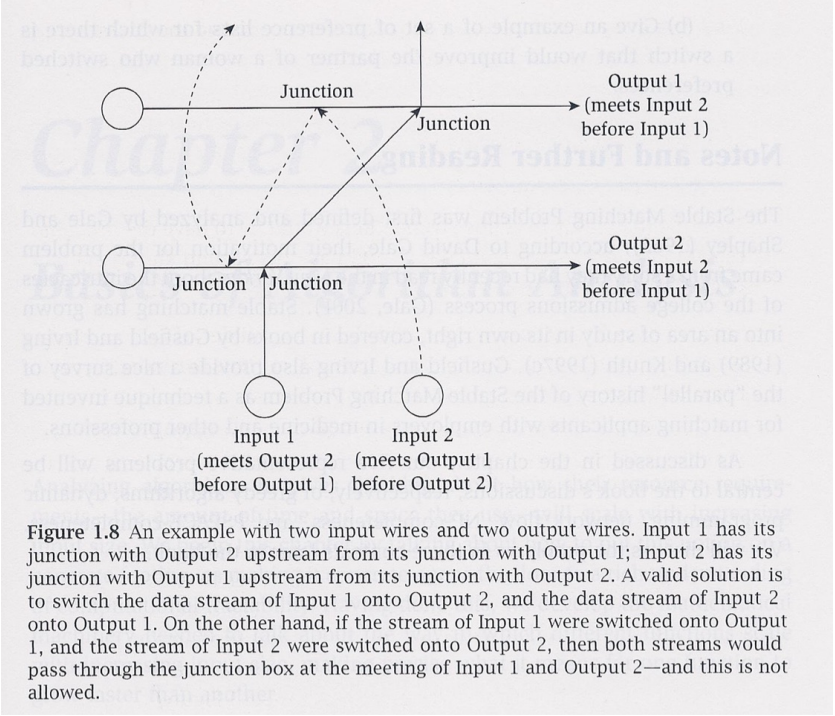

Figure 1.8 An example with two input wires and two output wires. Input 1 has its junction with Output 2 upstream from its junction with Output 1; Input 2 has its junction with Output 1 upstream from its junction with Output 2. A valid solution is to switch the data stream of Input 1 onto Output 2, and the data stream of Input 2 onto Output 1. On the other hand, if the stream of Input 1 were switched onto Output 1, and the stream of Input 2 were switched onto Output 2, then both streams would pass through the junction box at the meeting of Input 1 and Output 2—and this is not allowed.

Figure 1: From *Algorithm Design*, by Kleinberg and Tardos.

7. Prove that any connected undirected graph with n vertices and $n-1$ edges is a tree.

8. **(Ungraded Problem.)** Suppose you are the weekend scheduler for a major television network. Your superiors have collected opinions in an m-person poll, each person giving two possible movie titles that they want to see this weekend. Give an $O(m)$ algorithm that determines whether there exists a schedule, i.e. an assignment of movies to days (Saturday or Sunday), in which each person can see both of their desired movies between Saturday and Sunday and no one has to watch both their movies on the same day. A movie may **not** be aired multiple times over the weekend. You may assume that the number of movies which can be aired in one day is unbounded.