Upper bound for harmonic series

Let $n+1$ be a power of 2.

$$
\begin{aligned}
H_n &= \underbrace{1} + \underbrace{\frac{1}{2} + \frac{1}{3}} + \underbrace{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}} + \underbrace{\frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{15}} + \cdots + \underbrace{\frac{1}{(n+1)/2} + \cdots + \frac{1}{n-1} + \frac{1}{n}} \\
&\le \underbrace{1} + \underbrace{\frac{1}{2} + \frac{1}{2}} + \underbrace{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots + \frac{1}{8}} + \cdots + \underbrace{\frac{1}{(n+1)/2} + \cdots + \frac{1}{(n+1)/2} + \frac{1}{(n+1)/2}} \\
&= \underbrace{\underbrace{1} + \underbrace{1} + \underbrace{1} + \underbrace{1} + \cdots + \underbrace{1}}_{k \text{ times}} \\
&= k
\end{aligned}
$$

for some integer k.

The total number of terms n in the sum is

$$1 + 2 + 4 + \cdots + 2^{k-1}$$

So

$$n \;=\; \sum_{k=0}^{k-1} 2^k \;=\; 2^{(k-1)+1} - 1 \;=\; 2^k - 1$$

Adding 1 and then taking lg of both sides give

$$k \;=\; \lg(n+1)$$

In conclusion, $\lg(n+1)$ is an upper bound on the harmonic number H_n.