These are examples of proofs used in cmsc250. These proofs tend to be very detailed. You can be a little looser.

General Comments
Proofs by Mathematical Induction

- If a proof is by Weak Induction the Induction Hypothesis must reflect that. I.e., you may NOT write the Strong Induction Hypothesis.
- The Inductive Step MUST explicitly state where the Inductive Hypothesis is used. (Something like "by IH" is good.)

Example Proof by Weak Induction

Theorem. For $n \geq 1$, $\sum_{i=1}^{n} 4i - 2 \;=\; 2n^2$.

BASE CASE: Let $n = 1$. The summation gives

$$\sum_{i=1}^{n} 4i - 2 \;=\; \sum_{i=1}^{1} 4i - 2 \;=\; 4 \cdot 1 - 2 \;=\; 2 \;.$$

The formula gives

$$2n^2 \;=\; 2 \cdot 1^2 \;=\; 2 \;.$$

The two values are the same.

- **INDUCTIVE HYPOTHESIS** *[Choice I: From $n-1$ to n]:*

  Assume that the theorem holds for $n - 1$ (for arbitrary $n > 1$). Then

  $$\sum_{i=1}^{n-1} 4i - 2 \;=\; 2(n-1)^2 \;.$$

  [It is optional to simplify the right side. If not, it will have to be done inside the Induction Step.]

  - **INDUCTIVE STEP:** *[Choice Ia: Start with the sum we care about.]*

    $$\begin{array}{rcll} \sum_{i=1}^{n} 4i - 2 & = & \sum_{i=1}^{n-1} i \;\; + \;\; (4n-2) & \text{by splitting sum} \\ & = & 2(n-1)^2 \; + \; (4n-2) & \text{by IH} \\ & = & 2(n^2 - 2n + 1) \; + \; (4n-2) & \text{by algebra} \\ & = & 2n^2 \;. & \text{by algebra} \end{array}$$

    So the theorem holds for n.

  - **INDUCTIVE STEP:** *[Choice Ib: Start with the induction hypothesis.]*

    $$\begin{array}{rcll} \sum_{i=1}^{n-1} 4i - 2 & = & 2(n-1)^2 & \text{by IH} \\ \sum_{i=1}^{n-1} 4i - 2 \; + \; (4n-2) & = & 2(n-1)^2 \; + \; (4n-2) & \text{adding } 4n-2 \text{ to both sides} \\ \sum_{i=1}^{n} 4i - 2 & = & 2(n^2 - 2n + 1) \; + \; (4n-2) & \text{merging the sum on left side} \\ & & & \text{... and algebra on the right side} \\ & = & 2n^2 \;. & \text{by algebra on the right side} \end{array}$$

    So the theorem holds for n.

- **INDUCTIVE HYPOTHESIS:** *[Choice II: From n to $n+1$]*

  Assume that the theorem holds for arbitrary $n \geq 1$. Then

  $$\sum_{i=1}^{n} 4i - 2 \;=\; 2n^2 \;.$$

[The following is optional (but often useful). If you do not do this, you will struggle to make the right side have the correct form with n replaced by $n+1$. In principle, it should be inside the Inductive Step. At the very least, it must be clearly distinct from the Inductive Hypothesis.]

NEED TO SHOW:

$$\sum_{i=1}^{n+1} 4i - 2 \ = \ 2(n+1)^2 \ = \ 2(n^2+2n+1) \ = \ 2n^2+4n+2 \ .$$

– **INDUCTIVE STEP:** *[Choice IIa: Start with the sum we care about.]*

$$\begin{array}{rcll} \sum_{i=1}^{n+1} 4i-2 & = & \sum_{i=1}^{n} i \ + \ (4(n+1)-2) & \text{by splitting sum} \\ & = & 2n^2 \ + \ (4(n+1)-2) & \text{by IH} \\ & = & 2n^2 \ + \ (4n+2) & \text{by algebra} \\ & = & 2n^2+4n+2 \ . & \text{by algebra} \end{array}$$

This is what we needed to prove, so the theorem holds for $n+1$.

– **INDUCTIVE STEP:** *[Choice IIb: Start with the induction hypothesis.]*

$$\begin{array}{rcll} \sum_{i=1}^{n} 4i-2 & = & 2n^2 & \text{by IH} \\ \sum_{i=1}^{n} 4i-2 \ + \ (4(n+1)-2) & = & 2n^2 \ + \ (4(n+1)-2) & \text{adding } 4(n+1)-2 \text{ to both sides} \\ \sum_{i=1}^{n+1} 4i-2 & = & 2n^2+4n+2 & \text{merging sum on the left side} \\ & & & \text{... and algebra on the right side} \end{array}$$

This is what we needed to prove, so the theorem holds for $n+1$.

Example Proof by Strong Induction

BASE CASE: *[Same as for Weak Induction.]*

- **INDUCTIVE HYPOTHESIS:** *[Choice I: Assume true for less than n]*

  (Assume that for arbitrary $n > 1$, the theorem holds for all k such that $1 \le k \le n-1$.)

  Assume that for arbitrary $n > 1$, for all k such that $1 \le k \le n-1$ that

  $$\sum_{i=1}^{k} 4i - 2 \ = \ 2k^2 \ .$$

- **INDUCTIVE HYPOTHESIS:** *[Choice II: Assume true for less than $n+1$]*

  (Assume that for arbitrary $n \ge 1$ the theorem holds for all k such that $1 \le k \le n$.)

  Assume that for arbitrary $n > 1$, for all k such that $1 \le k \le n$ that

  $$\sum_{i=1}^{k} 4i - 2 \ = \ 2k^2 \ .$$

INDUCTIVE STEP: *[And now a brilliant proof that somehow uses strong induction.]*

Constructive Induction

[We do this proof only one way, but any of the styles is fine.]

Guess that the answer is quadratic, so it has form $an^2 + bn + c$. We will derive the constants a, b, c while proving it by Mathematical Induction.

BASE CASE: Let $n = 1$. The summation gives

$$\sum_{i=1}^{n} 4i - 2 \;=\; \sum_{i=1}^{1} 4i - 2 \;=\; 4 \cdot 1 - 2 \;=\; 2\;.$$

The formula gives

$$an^2 + bn + c \;=\; a \cdot 1^2 + b \cdot 1 + c \;=\; a + b + c\;.$$

So, we need $a + b + c = 2$.

INDUCTIVE HYPOTHESIS:
Assume that the theorem holds for $n - 1$ (for arbitrary $n > 1$). Then

$$\sum_{i=1}^{n-1} 4i - 2 \;=\; a(n-1)^2 + b(n-1) + c\;.$$

[Again, it is optional to simplify the right side.]

INDUCTIVE STEP:

$$\begin{array}{rcll}
\sum_{i=1}^{n} 4i - 2 & = & \sum_{i=1}^{n-1} 4i - 2 \;\;+\;\; (4n-2) & \text{by splitting sum} \\
& = & a(n-1)^2 + b(n-1) + c \;+\; (4n-2) & \text{by IH} \\
& = & a(n^2 - 2n + 1) + b(n-1) + c \;+\; (4n-2) & \text{by algebra} \\
& = & an^2 \;+\; (-2a + b + 4)n \;+\; a - b + c - 2 & \text{by algebra} \\
& = & an^2 + bn + c\;. & \text{to make the induction work}
\end{array}$$

The coefficients on each of the powers have to match. This leads to the three simultaneous equations:

$$\begin{array}{rcll}
b & = & -2a + b + 4 & \\
c & = & a - b + c - 2 & \\
2 & = & a + b + c & \text{from the base case}
\end{array}$$

The first equation gives $a = 2$, then the second gives $b = 0$, and finally the third gives $c = 0$.

Constructive Induction (Another Example)

Problem: Find an upper bound on F_n in the recurrence

$$F_n = F_{n-1} + F_{n-2}$$

where $F_0 = F_1 = 1$.

Guess that the answer is exponential, so $F_n \le ab^n$. We will derive the constants a, b while proving it by Mathematical Induction.

BASE CASES: Let $n = 0$. By definition

$$F_n = F_0 = 1$$

The formula gives

$$F_n \le ab^n = ab^0 = a$$

So, $a \ge 1$.

Let $n = 1$. By definition

$$F_n = F_1 = 1$$

The formula gives

$$F_n \le ab^n = ab^1 = ab$$

So, $ab \ge 1$.

INDUCTIVE HYPOTHESIS:
Assume that for arbitrary $n > 1$, for all k such that $1 \le k \le n-1$ that $F_k \le ab^k$.

INDUCTIVE STEP:

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} && \text{by definition} \\ &\le ab^{n-1} + ab^{n-2} && \text{by IH} \\ &\le ab^n \;. && \text{to make the induction work} \end{aligned}$$

Thus we need to solve

$$ab^{n-1} + ab^{n-2} \;\le\; ab^n \;.$$

or

$$b^2 - b - 1 \;\ge\; 0 \;.$$

By the quadratic formuls, we get

$$b \;\ge\; \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} \;=\; \frac{1 \pm \sqrt{5}}{2}$$

Only the positive value can hold. Also, we would like the smallest possible value for b. So we choose

$$b \;=\; \frac{1 + \sqrt{5}}{2}$$

From the base cases we get $a \ge 1$ (since the other condition is weaker), and now we would like the smallest possible value for a. So we choose $a = 1$. This gives

$$F_n \;\le\; \left(\frac{1 + \sqrt{5}}{2}\right)^n$$

Catalan Numbers

Theorem. *For* $n \geq 1$, $\frac{(2n)!}{n!(n+1)!} \geq \frac{4^n}{(n+1)^2}$.

Proof. by Mathematical Induction.

BASE CASE: Easy.

INDUCTION HYPOTHESIS: Assume true for $n-1$:

$$\frac{(2(n-1))!}{(n-1)!n!} \geq \frac{4^{n-1}}{n^2} .$$

INDUCTION STEP: *Alternative I*

$$\begin{aligned}
\frac{(2n)!}{n!(n+1)!} &= \frac{(2n)(2n-1)}{(n-1)n} \frac{(2(n-1))!}{(n-1)!n!} \\
&\geq \frac{(2n)(2n-1)}{n(n+1)} \frac{4^{n-1}}{n^2} && \text{by IH} \\
&= \frac{(2n)(2n-1)}{n(n+1)} \frac{(n+1)^2}{4n^2} \frac{4n^2}{(n+1)^2} \frac{4^{n-1}}{n^2} \\
&= \frac{(2n)(2n-1)}{(n-1)n} \frac{(n+1)^2}{4n^2} \frac{4^n}{(n+1)^2} \\
&= \frac{(1-1/2n)(1+1/n)}{1-1/n} \frac{4^n}{(n+1)^2} \\
&= \frac{1+1/(2n)-1/(2n^2)}{1-1/n} \frac{4^n}{(n+1)^2} \\
&\geq \frac{4^n}{(n+1)^2} .
\end{aligned}$$

INDUCTION STEP: *Alternative II*

$$\begin{aligned}
\frac{4^n}{(n+1)^2} &= \frac{4n^2}{(n+1)^2} \frac{4^{n-1}}{n^2} \\
&\leq \frac{4n^2}{(n+1)^2} \frac{(2(n-1))!}{(n-1)!n!} && \text{by IH} \\
&= \frac{4n^2}{(n+1)^2} \frac{n(n+1)}{(2n)(2n-1)} \frac{(2n)(2n-1)}{n(n+1)} \frac{(2n-1))!}{(n-1)!n!} \\
&= \frac{1}{(1+1/n)(1-1/(2n))} \frac{(2n)!}{n!(n+1)!} \\
&= \frac{1}{(1+1/(2n)-1/(2n^2))} \frac{(2n)!}{n!(n+1)!} \\
&\leq \frac{(2n)!}{n!(n+1)!} .
\end{aligned}$$

□