Problem 1. (25 points, 3-8 points each) Short answer questions. Explanations are not required, but may be given for partial credit.

(a) Give two examples that might arise in a game implemented in Unity, one where you want a rigid-body to have a collider and one where you want a trigger.

(b) You have three points $p$, $q$, and $r$ in the plane. You want to compute a point that lies close to the center of this triangle (I don’t care exactly where). Explain how to compute such a point using the operations of affine geometry.

(c) Consider the four points $a = (1,3)$, $b = (3,3)$, $c = (3,1)$ and $d = (1,1)$ in the figure below. List these points in Morton order (and briefly how you got your answer).

(d) In Unity, you want to rotate an object through $270^\circ$ degrees to occur smoothly over a period of 3 seconds. In your Update function, how many degrees of rotation should you apply? (Hint: Use the value of Time.deltaTime.)

(e) From the perspective of performance (time and/or space) list one advantage and one disadvantage of using a hashmap rather than an 3-dimensional array to represent a grid decomposition of 3-dimensional space.

Problem 2. (25 points) The objective of this problem is to derive a test for a cylindrical collider. The collider is defined by four parameters (see Figure 1(a)):

- the center point $p = (p_x, p_y, p_z)$ of the collider
- a unit-length vector $\vec{u} = (u_x, u_y, u_z)$ that points along the central axis of the cylinder
- a positive real $r$ that indicates the radius of the cylinder (perpendicular to the central axis)
- a positive real $\ell$ that indicates the length of the cylinder along its central axis
Figure 1: Cylinder Collider.

Our objective is to derive a procedure that will determine whether a given point \( q = (q_x, q_y, q_z) \) lies within the collider (see Figure 1(b)).

(a) (5 points) Given the points \( p \) and \( q \), show (using mathematical notation) how to compute the coordinates of a vector \( \vec{v} = (v_x, v_y, v_z) \) that is directed from \( p \) to \( q \) (see the figure (b)).

(b) (10 points) Given your answer to (a), show (using mathematical notation) how to decompose \( \vec{v} \) as the sum of two vectors \( \vec{v}' \) and \( \vec{v}'' \) such that \( \vec{v}' \) is parallel to \( \vec{u} \) and \( \vec{v}'' \) is perpendicular to \( \vec{u} \) (see Figure 1(c)). (Hint: Use the dot product.)

(c) (10 points) Given your answer to (b), show (using mathematical notation) how to compute the lengths of the vectors \( \vec{v}' \) and \( \vec{v}'' \) and then use these lengths together with \( r \) and \( \ell \) to determine whether \( q \) lies within the cylinder collider.

Problem 3. (30 points, 5–10 points each) Your company’s latest game involves a water cannon, which is used to extinguish fires in burning buildings. We will consider the problem in 2-dimensional space. The cannon’s bind pose is shown in Fig. 2(a). It consists of three rotatable joints: the base, the elbow, and the barrel. Water comes out from the nozzle point \( p \).

- Joint a (base joint) is at the origin
- Joint b (elbow joint) is 20 units above the origin
- Joint c (barrel joint) is 12 units to the right of the elbow joint
- Point p (nozzle) is 5 units to the right of the barrel joint
Given the three joint angles $\theta_a$, $\theta_b$, and $\theta_c$, we want to determine the location of nozzle point $p'$ (see Fig. 2(b)).

(a) What are the coordinates of the nozzle point $p$ in the bind pose relative to each of the following coordinate systems? Express each answer as a 3-element homogeneous vector:

(i) Barrel frame: $p_c = \ldots$
(ii) Elbow frame: $p_b = \ldots$
(iii) Base frame: $p_a = \ldots$

(b) Express the following local-pose transformations as homogeneous $3 \times 3$ matrices. (In all cases assume the bind pose shown in Fig. 2(a).)

(i) $T_{b\leftarrow c}$ (barrel-frame coordinates to the elbow-frame coordinates)
(ii) $T_{a\leftarrow b}$ (elbow-frame coordinates to the base-frame coordinates)

(c) What is the transformation $T_{a\leftarrow c}$ (barrel-frame coordinates to base-frame coordinates)? You may give your answer as a single $3 \times 3$ matrix or the product of matrices.

(d) Express the following inverse local-pose transformations as homogeneous $3 \times 3$ matrices (again, assuming the bind pose shown in Fig. 2(a).)

(i) $T_{c\leftarrow b}$ (elbow-frame coordinates to the barrel-frame coordinates)
(ii) $T_{b\leftarrow a}$ (base-frame coordinates to the elbow-frame coordinates)

(e) Suppose that we apply a rotation by angle $\theta_a$ about the base joint, $\theta_b$ about the elbow joint, and $\theta_c$ about the barrel joint. Let $\text{Rot}(\theta)$ denote a $3 \times 3$ homogeneous rotation matrix, that is

$$\text{Rot}(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Present a formula (as the product of matrices) that maps $p$ in the bind pose to its position $p'$ as a result of the rotations. Assume that $p$ and $p'$ are both represented relative to the base frame. That is, present a matrix $M$ (as the product of matrices) such that $p'_a = M p_a$. (Hint: It will be faster for you and easier for me if you express your matrices by name, e.g. “$T_{b\leftarrow c}$” rather than as a $3 \times 3$ matrix.)

Do only one of problems 4 or 5.

**Problem 4.** (20 points) Extending the water-cannon problem, we want to develop a targeting tool that determines where the water will hit a vertical wall. Suppose that the nozzle point of the water cannon is located $h$ units above the ground, and the water is being shot with velocity given by the vector $\vec{v}_0 = (v_{0,x}, v_{0,y})$. The wall is located $\ell$ units in front of the cannon (see Fig. 3).
Suppose we turn on the water at time $t = 0$. After consulting a standard textbook on Physics, we are reminded that gravity results in an acceleration of $g \approx 9.8 \text{m/s}^2$, and after $t$ time units have elapsed, the position of a projectile shot at velocity $\vec{v}_0$ is given by $p(t) = (x(t), y(t))$, where

$$x(t) = v_{0,x}t \quad \text{and} \quad y(t) = h + v_{0,y}t - \frac{1}{2}gt^2.$$ 

As a function of $h$, $\ell$, $g$, and $\vec{v}_0$, explain how to compute the height $y^*$ at which the water hits the wall. You may assume that the velocity is high enough that the water will reach the wall. (Hint: Start by computing the time it takes to reach the wall.)

**Problem 5.** (20 points) Suppose that we wanted to perform a rotation of $\theta = 60^\circ$ degrees about a unit vector $\vec{u} = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ using a quaternion representation (see Fig. 4).

(a) As a function of $\vec{u}$ and $\theta$, express this rotation as a unit quaternion $q$. (You may express $q$ as a 4-element vector or in the form $(s, \vec{u})$, where $s$ is a scalar and $\vec{u}$ is a 3-element vector.) Recall that

$$\sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos 60^\circ = \sin 30^\circ = \frac{1}{2}.$$ 

(These are the only trig values you might need.)

(b) What is the product of the following two quaternions? $q_1 = (1, 2, 0, 0) = 1 + 2i$ and $q_2 = (0, 3, 4, 0) = 3i + 4j$. Recall the rules of quaternion multiplication:

$$i^2 = j^2 = k^2 = ijk = -1 \quad \text{and} \quad ij = k, \; jk = i, \; ki = j.$$