Let G be undirected.

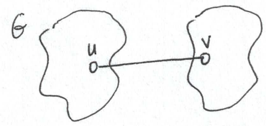
Menger's Thm: Graph G is 2-edge connected iff $\forall u,v \in V$ $(u \neq v)$, $\exists$ two edge-disjoint paths from u to v.

Proof: ($\Leftarrow$:) Suppose G is not 2-edge connected.

Case 1: G is not connected. Let u and v be in different connected components.

u and v do not have two edge-disjoint paths between them.

Case 2: G is connected. Then G contains a bridge $e=(u,v)$, i.e. removing e disconnects G.



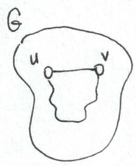
Then u and v cannot have two edge-disjoint paths between them.

($\Rightarrow$:) Suppose G is 2-edge connected (2EC). We need to show: $\forall u,v$ $(u \neq v)$ $\exists$ 2 edge-disjoint paths from u to v.

Will show by induction on distance $dist(u,v)$:

- Base case: for any u,v s.t. $dist(u,v)=1$, there must exist an edge $e=(u,v)$.

Since G is 2EC, e is not a bridge, i.e. $G \setminus \{e\}$ is still connected.



Let path $P' = \{e\}$.
Let path P'' be a path from u to v in $G \setminus \{e\}$.
P' and P'' are edge-disjoint.

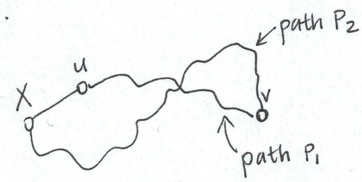
- Ind. Hyp.: Suppose $\forall u \neq v$ s.t. $dist(u,v) < k$, $\exists$ two edge-disjoint paths from u to v.
- Ind. Step: (Need to show $\forall u \neq v$ s.t. $dist(u,v) = k$, $\exists$ two edge-disjoint paths from u to v.)

Fix u,v s.t. $dist(u,v)=k$. Let X be the first node on a shortest path from u to v.

(In particular, $dist(X,v) < k$.)

- By the inductive hypothesis, and fact that $dist(X,v)<k$, $\exists$ two edge-disjoint paths P_1 and P_2 from X to v.
WLOG, P_1 doesn't visit any node more than once. Neither does P_2.

- Case 1: edge (u,X) is contained on one of the paths, say P_1.

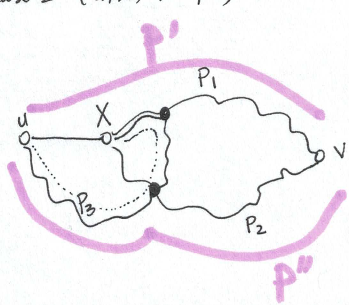


Then $\exists$ two edge-disjoint paths from u to v:

$P' = P_1 \setminus \{(X,u)\}$

$P'' = P_2 \cup \{(X,u)\}$

- Case 2: $(u,X) \notin P_1$, $(u,X) \notin P_2$.



- Let P_3 be a path from u to X that doesn't contain (u,X). (P_3 may share edges with P_1 & P_2, just not with (u,X).)
- Construct P' and P'' as follows:

P': follow P_3 out of u until it first hits a node on either P_1 or P_2, WLOG P_2. (X is on P_1 and P_2.) Hop onto P_2 and follow it to v.

P'': follow (u,X), then hop onto P_1 and follow it to v.

To see that P' and P'' are edge-disjoint:

- $P' \cap P_3$ (i.e. first part of P') doesn't intersect (u,X) by edge-disjointedness of P_3 and (u,X), and doesn't intersect P_1 by the fact that P_3 hit P_2 first.
- $P' \cap P_2$ (i.e. second part of P') doesn't intersect (u,X) by fact that we're in Case 2, and doesn't intersect P_1 by edge-disjointedness of P_1 and P_2.

$\therefore$ P' and P'' are edge-disjoint.

$\blacksquare$