Assume that you are interested in the average number of exchanges in Bubble Sort and its distribution. (As usual, assume that the elements are distinct, and that each permutation is equally likely.)

Problem 1. We start with the average number of exchanges in the *first iteration* of Bubble Sort on a list of size n. Here is the code:

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for j = 1 to n-1 do if A[j] > A[j+1] then A[j] ↔ A[j+1]
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It turns out that on average most potential exchanges will actually be made, so it is mildly easier to count the number of times an exchange is *not* made. (A "potential exchange" means that a comparison was made that could have resulted in an exchange. So you can think of a potential exchange as being a comparison, or you can think of the number of potential exchanges as being the worst case number of exchanges.)

(a) How many potential exchanges are there in the first iteration (of size n) of Bubble Sort?

(b) Briefly explain why on average most potential exchanges will be made in the first iteration of Bubble Sort.

(c) Calculate the exact average number of potential exchanges that are not made in the first iteration of Bubble Sort. Your answer should use the harmonic series (H_n).

(d) What is the exact average number of exchanges that *are* made in the first iteration of Bubble Sort. Your answer should use the harmonic series (H_n).

(e) What is the above formula without using the harmonic series? Give the exact first and second order terms.

Problem 2. We now "analyze" the average number of exchanges in the full Bubble Sort on a list of size n. Make the simplifying (but wrong) assumption that every iteration starts with a random permutation.

(a) Give a summation for the exact average number of exchanges (on a list of size n).

(b) Simplify the summation. You can either calculate this by hand or look it up. One good resource is Wolfram Alpha. Give the answer as a function of H_n (not H_{n+1}).

(c) What is the average number of exchanges in Bubble Sort? Give the exact first and second order terms without the harmonic series.

Problem 3. We now analyze the distribution of the number of exchanges in the *first iteration* of Bubble Sort. In other words, how often the first iteration of Bubble Sort does no exchanges, how often it does one exchange, how often it does two exchanges, etc. Assume that the list has size n. Once again, it is easier to count the number of times a potential exchange is *not* made.

(a) For each value of n from 1 to 4, consider all permutations of n elements and count how many potential exchanges are not made. Fill in the first column of the two tables (at the end of the assignment) for $n = 3$ and $n = 4$.

(b) Fill in the following table, leaving blank any value that cannot occur. (You can just give your answer as four separate lists.)

Size n	Number of permutations with given number of potential exchanges not made			
	0	1	2	3
1				
2				
3				
4				

(c) What are these values? If you happen not to recognize them, a good resource is the On-Line Encyclopedia of Integer Sequences. Just the last row of the table should be enough.

(d) Fortunately there are standard notations for these values. Let $P(n, i)$ be the probability that i potential exchanges are *not* made on a list of size n in the *first iteration* of Bubble Sort. Using the standard notation with square brackets, give a simple formula for $P(n, i)$.

(e) Let $Q(n, i)$ be the probability that i exchanges *are* made on a list of size n in the *first iteration* of Bubble Sort. Using the standard notation with square brackets, give a simple formula for $Q(n, i)$.

(f) CHALLENGE PROBLEM (will not be graded): What we have is strong evidence that we know the distribution, but it is not a proof. Prove that your distribution function is correct.

Problem 4. We now analyze the distribution of the number of exchanges in the full Bubble Sort on a list has size n.

(a) For each value of n from 1 to 4, consider all permutations of n elements and count how many exchanges are made. Fill in the second column of the two tables (at the end of the assignment) for $n = 3$ and $n = 4$.

(b) Fill in the following table, leaving blank any value that cannot occur. (You can just give your answer as four separate lists.)

Size n	Number of permutations with given number of exchanges						
	0	1	2	3	4	5	6
1							
2							
3							
4							

(c) What are these values? Once again, if you happen not to recognize them, a good resource is the On-Line Encyclopedia of Integer Sequences. Just the last row of the table should be enough.

$n = 3$	First iteration	Full Bubble Sort
	Number of potential exchanges not made	Number of exchanges made
123		
132		
213		
231		
312		
321		

$n = 4$	First iteration	Full Bubble Sort
	Number of potential exchanges not made	Number of exchanges made
1234		
1243		
1324		
1342		
1423		
1432		
2134		
2143		
2314		
2341		
2413		
2431		
3124		
3142		
3214		
3241		
3412		
3421		
4123		
4132		
4213		
4231		
4312		
4321		