CMSC 425 : Spring 2018 Dave Mount and Roger Eastman

Practice Problems for the First Midterm

The midterm will be given in class on **Wed, April 4** for Eastman's section and **Thu, April 5** for Mount's section. The exam will be closed-book and closed-notes, but you will be allowed one sheet of notes (front and back).

Disclaimer: These are practice problems, which have been drawn from old homeworks and exams. They **do not** reflect the actual length, difficulty, or coverage for the exam.

Problem 1. Short answer questions. Unless otherwise specified, explanations are not required, but may be provided for partial credit.

(a) Suppose that a Unity game object is declared to be *static* (by checking the "Static" checkbox in the editor). Which of the following optimizations can Unity perform as a result? (Indicate True or False for each.)

(i) Navigation computation and physics can be optimized because the object's position is fixed.

(ii) Fewer method calls are needed, because the methods Update or FixedUpdate are never called on static objects.

(iii) Some global lighting computations can be precomputed.

(iv) Space is saved because all instantiations of a static object refer to the same (shared) game object.

(b) Consider an orthonormal coordinate frame in 3-space, where $\vec{u}_x$, $\vec{u}_y$, and $\vec{u}_z$ are the unit vectors. What relationship holds between these vectors in order for the frame to be *right handed*? (You can express your answer as an equation involving geometric operations or as a drawing.)

(c) You have a long, thin object (e.g., an arrow) that can be oriented arbitrarily in space. Which of the following collider shapes would NOT be a good choice to represent this object (Select all the apply). Briefly *explain* your answers.

(i) Axis-aligned bounding box (AABB)

(ii) General (arbitrarily oriented) bounding box

(iii) Bounding sphere

(iv) Capsule

(d) Consider the following two computational tasks that arise in animation processing:

Task I: Given the placement of a skeletal model in a scene and an assignment to its joint angles, determine the position of a point of the model (e.g., the tip of the index finger) relative to the scene's coordinate system.

Task II: Given the placement of a skeletal model in a scene and the desired position of a given point of the model (e.g., the tip of the index finger should be touching a light switch), determine how to set the joint angles to achieve this desired result.

(i) One of the above tasks is an instance of *forward kinematics* and the other is an instance of *inverse kinematics*. Which is which?

(ii) Which of these two tasks is computationally more challenging? Briefly justify your answer.

(e) Let S^* denote the limiting shape resulting from the sequence of curves shown in Fig. 1. What is the fractal dimension of this object? (You may express your answer as the ratio of logarithms.)

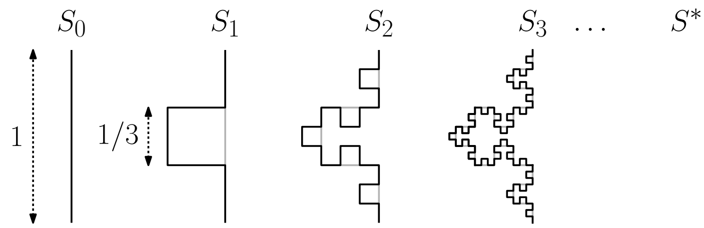


Figure 1: Problem 1(e): Fractal dimension.

Problem 2. Your new 3-dimensional game involves throwing a Frisbee. You need to implement an efficient collider that will (roughly) represent the shape of a flying disk. You have chosen to model the Frisbee collider as a simple flat circular disk in three dimensional space (with zero thickness). The collider is specified by three parameters: (1) the center point $p = (p_x, p_y, p_z)$ of the collider, (2) a unit-length normal vector $\vec{u} = (u_x, u_y, u_z)$ that points in the direction perpendicular to the plane on which the disk lies, and (3) a positive real r that indicates the radius of the disk (see Fig. 2(a)).

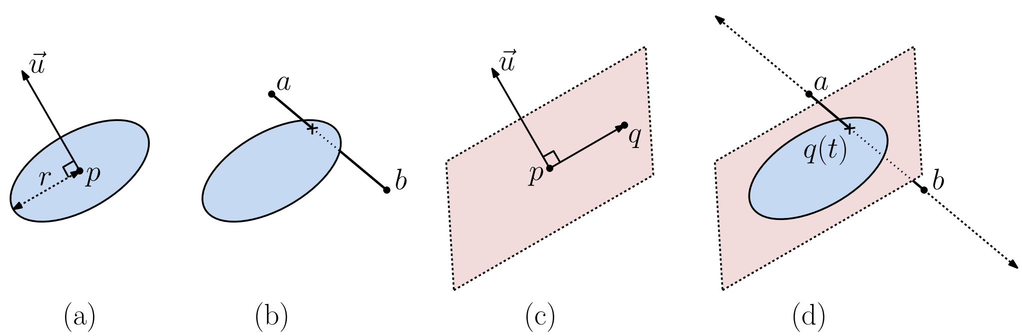


Figure 2: Problem 2: Frisbee collider.

The objective of this problem is to derive a procedure that, given a Frisbee collider $\langle p, \vec{u}, r \rangle$ and a line segment $\overline{ab}$, where $a = (a_x, a_y, a_z)$ and $b = (b_x, b_y, b_z)$, determines whether the Frisbee collider intersects the line segment (see Fig. 2(b)). You may assume that $a \neq b$ and neither of the points a or b lies on the plane that contains the collider.

(a) The first step is to determine the equation of the infinite plane containing the collider disk. A point $q = (x, y, z)$ lies on the plane if and only if the free vector directed from p to q is perpendicular to the vector $\vec{u}$ (see Fig. 2(c)). Use this fact to derive the equation of the plane. (Hint: The plane equation can be expressed in the form $\alpha x + \beta y + \gamma z + \delta = 0$ for some scalars α, β, γ, and δ. Derive the values of these four scalars as a function of the coordinates of p and $\vec{u}$.)

(b) We showed in class that any point on the infinite line $\overleftrightarrow{ab}$ can be expressed as the affine combination $(1 - t)a + tb$, for some real t. Using your answer from part (a), derive a procedure (in mathematical notation) for computing the value of t where the infinite line hits the collider plane. Let's call this point $q(t)$ (see Fig. 2(d)). Also, present a test to determine whether $q(t)$ lies within the (finite) line segment $\overline{ab}$.

(c) Your answer to (b) should involve division by a quantity that depends on the inputs. Under what conditions (as a function of $\vec{u}$, p, a, and/or b) would the divisor(s) be equal to zero? Does the problem description exclude this possibility? (If not, what additional assumptions need to be added?)

(d) Assuming that $q(t)$ (from part (b)) exists and lies within the line segment $\overline{ab}$, explain how to determine whether $q(t)$ lies within the collider disk of radius r (see Fig. 2(d)).

Problem 3. The objective of this problem is to derive a test for a cylindrical collider. The collider is defined by four parameters (see Fig. 3(a)):

- the center point $p = (p_x, p_y, p_z)$ of the collider
- a unit-length vector $\vec{u} = (u_x, u_y, u_z)$ that points along the central axis of the cylinder
- a positive real r that gives the radius of the cylinder (perpendicular to the central axis)
- a positive real ℓ that indicates the length of the cylinder along its central axis

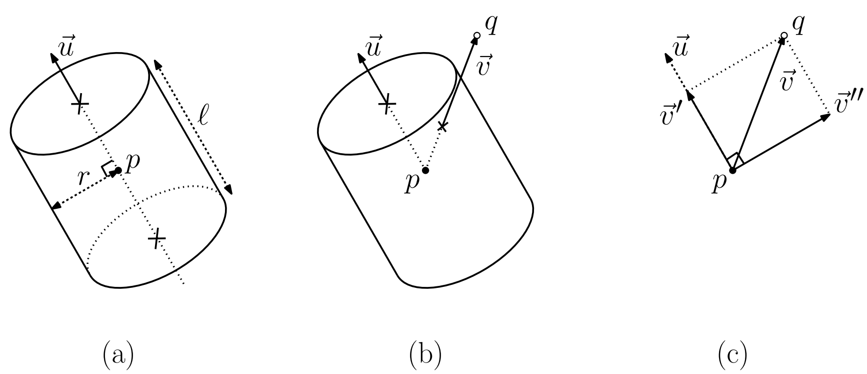


Figure 3: Problem 3: Cylinder collider.

Our objective is to derive a procedure that will determine whether a given point $q = (q_x, q_y, q_z)$ lies within the collider (see Fig. 3(b)).

(a) Given the points p and q, show (using mathematical notation) show how to compute the coordinates of a vector $\vec{v} = (v_x, v_y, v_z)$ that is directed from p to q (see the figure (b)).

(b) Given your answer to (a), show (using mathematical notation) how to decompose $\vec{v}$ as the sum of two vectors $\vec{v}'$ and $\vec{v}''$ such that $\vec{v}'$ is parallel to $\vec{u}$ and $\vec{v}''$ is perpendicular to $\vec{u}$ (see Fig. 3(c)).

(c) Given your answer to (b), show (using mathematical notation) how to compute the lengths of the vectors $\vec{v}'$ and $\vec{v}''$ and then use these lengths together with r and ℓ to determine whether q lies within the cylinder collider.

Problem 4. You are implementing a football game, and you want to simulate the process of a quarterback throwing the ball to a pass receiver. The receiver is running horizontally across the field at a fixed speed of s_p feet per second, and the quarterback throws the ball at a fixed speed of s_q feet per second. (You may assume that both of these quantities are positive.) The quarterback needs to adjust the angle at which the ball is thrown (thus, leading the receiver) so that the ball arrives at the same time as the receiver.

To simplify matters, let us do this in the 2-dimensional plane. Assume that the quarterback is located at a point $q = (q_x, q_y)$, and at the instant the ball is thrown the receiver is at point $p = (p_x, p_y)$ directly above q. Thus, $p_x = q_x$ and $p_y > q_y$ (see Fig. 4). Let $\ell_q = p_y - q_y$ denote the initial distance between the quarterback and receiver, and assume that the receiver moves horizontally to the right.

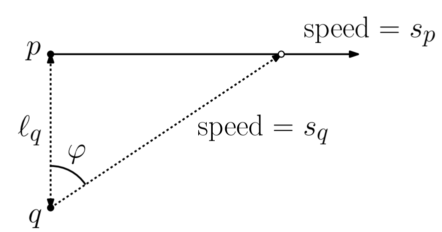


Figure 4: Problem 4: Throwing a football.

Derive (in mathematical notation) a procedure, which given q, p, s_q and s_p, outputs the angle $\varphi > 0$ of the direction (relative to the vector from q to p) at which the quarterback should throw the ball so that the receiver and ball arrive at the same time in the same place (assuming that they move at their given speeds). You may express φ either in radians or degrees.

In order for your solution to exist, what assumptions need to be made about the relationship between s_q and s_p?

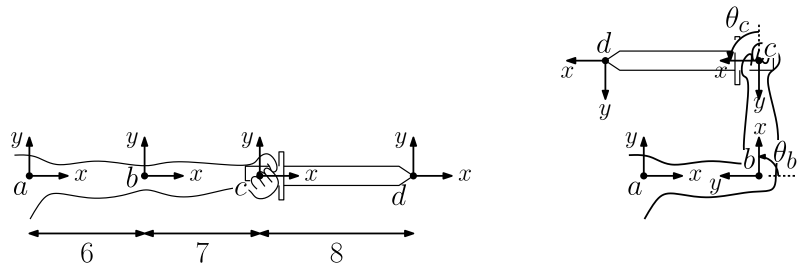
Problem 5. Consider a skeletal model of an arm holding a sword in 2-dimensional space. Suppose that the bind pose is as shown in Fig. 5(a), with the arm and sword extending horizontally to the right of the shoulder. The shoulder, elbow, hand, and tip of sword coordinate frames are called a, b, c, and d, respectively. It is 6 units from the shoulder to the elbow, 7 units from the elbow to the hand, and 8 units from the hand to the tip of the sword.

(a) Following the naming convention for the local pose transformations (given in Lecture 9) express the following local pose transformations a 3×3 homogeneous matrices. (In all cases assume the arrangement shown in Fig. 5(a).)

(i) $T_{[c \leftarrow d]}$, which translates coordinates in the sword tip frame to the hand frame.

(ii) $T_{[b \leftarrow c]}$, which translates coordinates in the hand frame to the elbow frame.

(iii) $T_{[a \leftarrow b]}$, which translates coordinates in the elbow frame to the shoulder frame.



(a) (b)

Figure 5: Problem 5: Kinematics for a skeletal arm.

For example, the transformation $T_{[c \leftarrow d]}$ should transform the column vector denoting the tip of the sword relative to the tip-of-sword frame coordinate (as the origin) to its representation relative to the hand frame coordinates (as lying 8 units along the x-axis). That is,

$$T_{[c \leftarrow d]} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 1 \end{pmatrix}.$$

(b) Show that by multiplying these matrices together in the proper order, we obtain a matrix $T_{[a \leftarrow d]}$ that maps a point in the tip-of-sword frame to the shoulder frame. For example, because the tip lies 21 units to the right of the should, we have

$$T_{[a \leftarrow d]} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 21 \\ 0 \\ 1 \end{pmatrix}.$$

(c) Give the following inverse local pose transformations:

(i) $T_{[d \leftarrow c]}$, which translates coordinates in the hand frame to the sword tip frame.

(ii) $T_{[c \leftarrow b]}$, which translates coordinates in the elbow frame to the hand frame.

(iii) $T_{[b \leftarrow a]}$, which translates coordinates in the shoulder frame to the elbow frame.

(Hint: You can exploit the simple structure of the matrices in part (a) to avoid the need for general matrix inversion.)

(d) Suppose that we apply a rotation by angle θ_b about the elbow and θ_c about the hand. (These are both $90° = \pi/2$ in Fig. 5(b), but they can be any angle, positive or negative, in general.) Assume that $\mathrm{Rot}(\theta)$ denotes a 3×3 rotation matrix, that is

$$\mathrm{Rot}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let's assume that all points are represented in the shoulder frame. Following the example in Lecture 9, derive a matrix (which you may express as the product of a sequence of matrices) that maps a point representing the tip of the sword in the bind pose to its rotated position. For example, in the particular case where $\theta_b = \theta_c = 90°$, this would map the vector $(21, 0, 1)$ to $(-2, 7, 1)$. (Your answer should work for any values of θ_b and θ_c.)

Explain how you derived your answer.

Problem 6. Your company's latest game involves a water cannon, which is used to extinguish fires in burning buildings. We will consider the problem in 2-dimensional space. The cannon's bind pose is shown in Fig. 6(a). It consists of three rotatable joints: the base, the elbow, and the barrel. Water comes out from the nozzle point p.

- Joint a (base joint) is at the origin
- Joint b (elbow joint) is 20 units above the origin
- Joint c (barrel joint) is 12 units to the right of the elbow joint
- Point p (nozzle) is 5 units to the right of the barrel joint

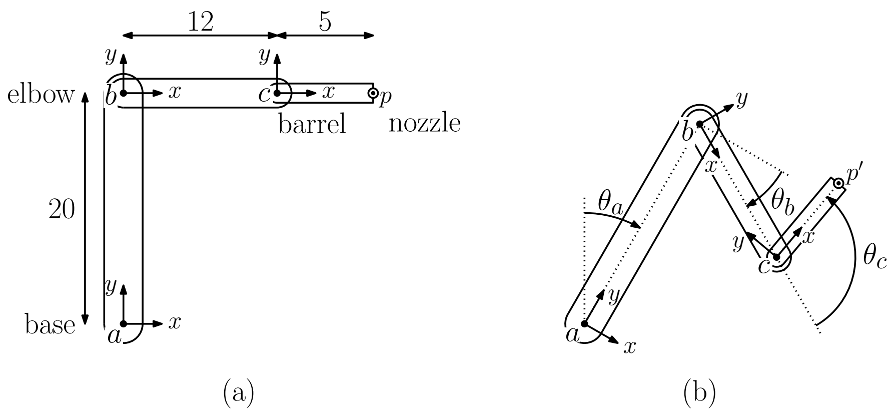


Figure 6: Problem 6: Kinematics for a water canon.

Given the three joint angles θ_a, θ_b, and θ_c, we want to determine the location of nozzle point p' (see Fig. 6(b)).

(a) What are the coordinates of the nozzle point p in the bind pose relative to each of the following coordinate systems? Express each answer as a 3-element homogeneous vector:

(i) Barrel frame: $p_{[c]} =$

(ii) Elbow frame: $p_{[b]} =$

(iii) Base frame: $p_{[a]} =$

(b) Express the following local-pose transformations as homogeneous 3×3 matrices. (In all cases assume the bind pose shown in Fig. 6(a).)

(i) $T_{[b \leftarrow c]}$ (barrel-frame coordinates to the elbow-frame coordinates)

(ii) $T_{[a \leftarrow b]}$ (elbow-frame coordinates to the base-frame coordinates)

(c) What is the transformation $T_{[a \leftarrow c]}$ (barrel-frame coordinates to base-frame coordinates)? You may give your answer as a single 3×3 matrix or the product of matrices.

(d) Express the following inverse local-pose transformations as homogeneous 3×3 matrices (again, assuming the bind pose shown in Fig. 6(a).)

  (i) $T_{[c \leftarrow b]}$ (elbow-frame coordinates to the barrel-frame coordinates)

  (ii) $T_{[b \leftarrow a]}$ (base-frame coordinates to the elbow-frame coordinates)

(e) Suppose that we apply a rotation by angle θ_a about the base joint, θ_b about the elbow joint, and θ_c about the barrel joint. Let $\mathrm{Rot}(\theta)$ denote a 3×3 homogeneous rotation matrix, that is

$$\mathrm{Rot}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Present a formula (as the product of matrices) that maps p in the bind pose to its position p' as a result of the rotations. Assume that p and p' are both represented relative to the *base frame*. That is, present a matrix M (as the product of matrices) such that $p'_{[a]} = M p_{[a]}$. (Hint: It will be faster for you and easier for me if you express your matrices by name, e.g. "$T_{[b \leftarrow c]}$" rather than as a 3×3 matrix.)

Problem 7. Extending the water-cannon problem, we want to develop a targeting tool that determines where the water will hit a vertical wall. Suppose that the nozzle point of the water cannon is located h units above the ground, and the water is being shot with velocity given by the vector $\vec{v}_0 = (v_{0,x}, v_{0,y})$. The wall is located ℓ units in front of the cannon (see Fig. 7).

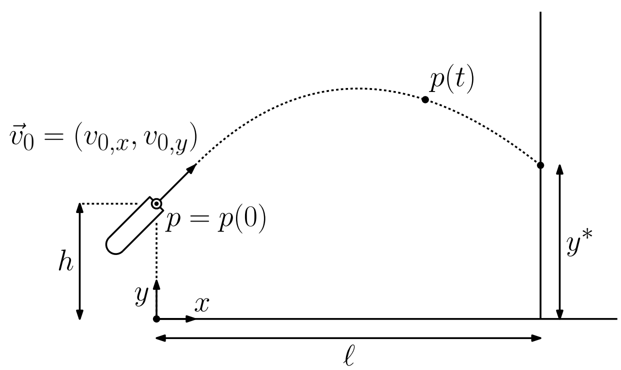


Figure 7: Problem 7: Water canon targeting tool.

Suppose we turn on the water at time $t = 0$. After consulting a standard textbook on Physics, we are reminded that gravity results in an acceleration of $g \approx 9.8 m/s^2$, and after t time units have elapsed, the position of a projectile shot at velocity $\vec{v}_0$ is given by $p(t) = (x(t), y(t))$, where

$$x(t) = v_{0,x} t \quad \text{and} \quad y(t) = h + v_{0,y} t - \frac{1}{2} g t^2.$$

As a function of h, ℓ, g, and $\vec{v}_0$, explain how to compute the height y^* at which the water hits the wall. You may assume that the velocity is high enough that the water will reach the wall. (Hint: Start by computing the time it takes to reach the wall.)

Problem 8. Suppose that we wanted to perform a rotation of $\theta = 60°$ degrees about a unit vector $\vec{u} = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ using a quaternion representation (see Fig. 8).

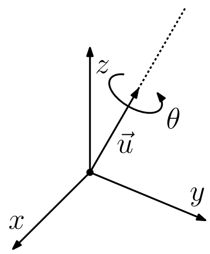


Figure 8: Problem 8: Quaternions.

(a) As a function of $\vec{u}$ and θ, express this rotation as a *unit quaternion* $\mathbf{q}$. (You may express $\mathbf{q}$ as a 4-element vector or in the form $(s, \vec{u})$, where s is a scalar and $\vec{u}$ is a 3-element vector.) Recall that

$$\sin 60^\circ \;=\; \cos 30^\circ \;=\; \frac{\sqrt{3}}{2} \qquad \text{and} \qquad \cos 60^\circ \;=\; \sin 30^\circ \;=\; \frac{1}{2}.$$

(These are the only trig values you might need.)

(b) What is the product of the following two quaternions? $\mathbf{q}_1 = (1, 2, 0, 0) = 1 + 2i$ and $\mathbf{q}_2 = (0, 3, 4, 0) = 3i + 4j$. Recall the rules of quaternion multiplication:

$$i^2 = j^2 = k^2 = ijk = -1 \qquad ij = k,\; jk = i,\; ki = j.$$