23 Appendix – Projective Geometry for Machine Vision

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...ὥστε καλλιον ἀποδεξεσθαι, ἴσμεν που ὅτι τῳ ὅλῳ και παντι διοισει ἡμμενος τε γεωμετραις και μη.

"... experience proves that anyone who has studied geometry is infinitely quicker to grasp difficult subjects than one who has not."
Plato – The Republic, Book 7, 375 B.C.

23.1 Introduction

The idea for this Appendix arose from our perception of a frustrating situation faced by vision researchers. For example, one is interested in some aspect of the theory of perspective image formation such as the epipolar line. The interested party goes to the library to check out a book on projective geometry filled with hope that the necessary mathematical machinery will be directly at hand. These expectations are quickly dashed. Upon opening the book, the expectant reader finds the presentation dominated by endless observations about harmonic relations and a few chapters which explore the minutiae of Pappus' theorem. Finally, as a last cruel twist of irony, the book ends in triumph with a rather exhilarating discourse on the conic pencil. All of the material is presented in the form of theorems defined on points, lines and conics without the use of coordinates, except perhaps for a quick pause to define barycentric coordinates just to taunt the reader. Dejected, the vision researcher throws the book aside and contents himself with some calculations using homogeneous coordinates and transformations which are covered briefly in Duda and Hart [93] or perhaps from a book on graphics [113].

A major reason for this state of affairs is that projective geometry is often formalized from the *synthetic* point of view. In the synthetic approach, points and lines are purely abstract predicates which are defined and related by axioms and theorems. The realization of these structures in everyday experience is of little importance to the mathematician. The important issue from a mathematical point of view is to set up the axiomatic structure of the projective plane and then explore the logical implications of the structure. The synthetic approach is a legitimate mathematical enterprise but leaves much to be desired for applications in vision research. In addition, many projective geometry books adopt a style of presentation which carries through a dense series of definitions, lemmas and proofs. Ultimately, these theorems can be of great importance to vision applications but the classical presentation style provides little or no guidance about the significance of each result.

Perhaps the most useful presentation for applications in vision is the *analytic* approach. Analytic projective geometry is quite analogous to analytic Euclidean geometry which is widely taught and forms the mainstay of first year calculus courses. The central concept of the analytic approach is to introduce coordinates right at the beginning and then define the geometric entities as equations on the coordinates. From the analytic viewpoint, a particular geometric configuration corresponds to a solution of the set of coordinate equations which define the entities and their relations. Fortunately, there are a few accessible books which pursue the analytic approach in projective geometry [270, 265]. These books, if carefully read and digested, provide a rich source of ideas in vision.

The main purpose of this Appendix is to compile many of the useful analytic concepts from projective geometry and demonstrate their relationship to vision problems. At the same time, it is hoped that the presentation will provide a convenient reference for many of the concepts used by the authors in the preceding chapters, particularly concerning the development of invariants.

23.2 Projective geometry in vision

23.2.1 The past

Perhaps the most important question to address at the beginning is the relevance of projective geometry in vision. Indeed, there may be some suspicion that projective geometry is not very relevant at all since most mathematics departments do not provide a course on projective geometry, even at the graduate level. The absence of adequate course offerings is more likely due to the fact that most of the interesting mathematical problems of classical projective geometry were solved in the 19th century, rather than a lack of practical relevance.

Projective geometry was used in vision almost right from the start. For example in 1965, Roberts [250] in his Ph.D. thesis used the 3D to 2D homogeneous projective transformation matrix to represent the imaging of 3D polyhedral objects. He developed a linear algorithm for determining the matrix parameters and was able to verify object models by projection onto the image plane.

In vision papers over the years, the properties of perspective projection have been derived in many different ways. Most treatments start by writing down the equations of central projection[1] and then continue with some simplifications and new grouping of parameters which constitute the main theoretical result. This approach has produced many

[1]The most common formulation is, $x = f\frac{X}{Z}, y = f\frac{Y}{Z}$, where (X, Y, Z) is a 3D point in world coordinates and (x, y) is the perspective image of the point. It is assumed that the camera coordinate axes are aligned with the world axes. The camera focal length is f. Note that the transformation is not linear since the image position is proportional to inverse depth.

important results but is not very powerful in attacking generic problems. An excellent example is the problem of determining the minimum number of point correspondences needed to compute the transformation between two perspective views of a 3D point set. The 3D locations of the points are not known in advance. Recently, a proof that five points are sufficient has been established by invoking the machinery of analytic projective geometry. It is unlikely that the insights obtained about this problem could have been achieved otherwise [107].

Perhaps the first attempt to encourage vision researchers to make use of general results in projective geometry was the textbook by Duda and Hart [93][2]. They devote several chapters to the concept of the cross-ratio and its potential uses in the analysis of epipolar geometry. For some reason, this material was not received with much interest. Perhaps the application suggestions made by Duda and Hart were not powerful enough or did not seem to attack the core problems in machine vision. Also, orthographic projection was considered adequate for most of the blocks world vision problems which were popular in the late 1960s, such as Huffman–Clowes labeling [302].

Subsequent vision texts [29, 157] do not consider results from projective geometry except to establish the relationship between the geometry of perspective viewing and the homogeneous transformation matrix. These equations are then exploited in applications such as stereo and photogrammetry.

23.2.2 Why do we need projective geometry?

The geometry of objects is strongly distorted by perspective image projection. The perspective transformation of geometric shapes cannot be accounted for by the usual mechanisms of Euclidean geometry. The main value of a mathematical framework is that the model should account for all the important phenomena with compact and easily manipulated structures. Under perspective projection, parallel lines do not remain parallel but instead meet at a point called the *vanishing point.* The convergence of parallel lines under perspective is illustrated in Figure 23.1 where a number of vanishing points are indicated. The fact that parallel lines always meet at a vanishing point is the main property which distinguishes projective geometry from Euclidean geometry. Indeed, the only geometric properties preserved under projective transformations are collinearity, tangency and incidence conditions, such as intersection and concurrence. This paucity of invariant geometric properties is a major contributor to the difficulty of object description and recognition under perspective viewing. For example, the SCERPO [201] system assumes affine projection in order to use parallelism as a perceptual grouping relation. Also, affine geometry is often assumed in model-based vision

[2] A call taken up by Naeve and Eklundh [216].

Figure 23.1
The concept of parallelism is not meaningful for perspective projection. Notice that parallel lines converge to a vanishing point at the horizon.

systems [56, 159] because fewer features are required to compute model pose uniquely under affine projection than perspective projection. However, the affine approximation to perspective fails when the depth range of an object is significant compared to the viewing distance.

The most important aspect of projective geometry is the introduction of homogeneous coordinates which represent a projective transformation as a matrix multiplication. This compact form allows many of the significant aspects of projective transformations and projective geometry to be demonstrated using simple matrix algebra computations. In Euclidean coordinates, many of these derivations become difficult, if not impossible.

We now illustrate the relevance of projective geometry by reviewing a few major vision problems which have been solved using results from projective geometry.

23.2.3 Contributions of projective geometry to vision

The projective transformation matrix – In his Ph.D. thesis, L. Roberts [250] developed a complete system for the recognition of polyhedral models in grey level images. One of the problems he solved was to derive a linear method for determining the projective transformation matrix for a camera, given a set of 3D points and their image locations. The solution found by Roberts[3] is quite attractive since it involves only matrix algebra and requires a minimum of six reference points. The method does not require an iterative solution and any number of points can be used in a least mean square sense to improve the accuracy of the camera parameters.

[3] see Section 23.10 for further discussion.

Plane orientation from vanishing points – A number of results have been achieved in computing the orientation of planes from vanishing points of parallel line sets in the plane [78]. Perhaps the most extensive investigation of the use of vanishing points in the implementation of a full system is by Torre and Coelho [77]. They restricted their experiments to an environment containing objects with parallel or perpendicular sides called Legoland. These assumptions allow recovery of the 3D geometry of a scene from vanishing points constructed in a single image of the scene.

The focus of expansion – When a camera is undergoing pure translational motion, it is well known that the trajectories of image points appear to move towards or away from a fixed point known as the Focus of Expansion (FOE). Projective geometry gives an immediate insight into this situation: relative to the camera all space points are moving along parallel straight lines. Clearly, the projections of these lines converge to a common vanishing point – the FOE [169].

Camera motion from n matched points – Once the transformation due to camera motion is known, the 3D position of the points can be recovered up to an unknown scale factor. This problem, which is important for many visual tasks, has been in the literature since Chasles [72] in 1855. The main focus has been on finding the minimum number of correspondences to solve for the motion (the minimum is five) and how many solutions are produced for a particular number of point correspondences. A few examples of recent investigations are:

1. Longuet-Higgins [198, 199] showed that for $n = 8$, a solution can be recovered using linear techniques[4];

2. Maybank [204] showed that there are multiple solutions (for any n) if points and optical centers lie on a certain critical surface (a ruled quadric);

3. Faugeras and Maybank [107] proved that for $n = 5$ there are at most 10 solutions.

To determine these results the authors made elegant and very sophisticated use of projective geometry. However, as has been pointed out by Buchanan [60], some of these modern developments are a rediscovery of results known to projective geometers and photogrammetrists of the 19th century [111, 147, 186, 277]. The main reason that these earlier results were rediscovered is best summed up by a quote by Kanatani, "the works are inaccessible and illegible to modern readers." Again, the inaccessibility of the 19th century results underscores the need for the migration of this literature into modern notation.

[4] See Section 23.11 for more detail on the analysis of two corresponding projective views.

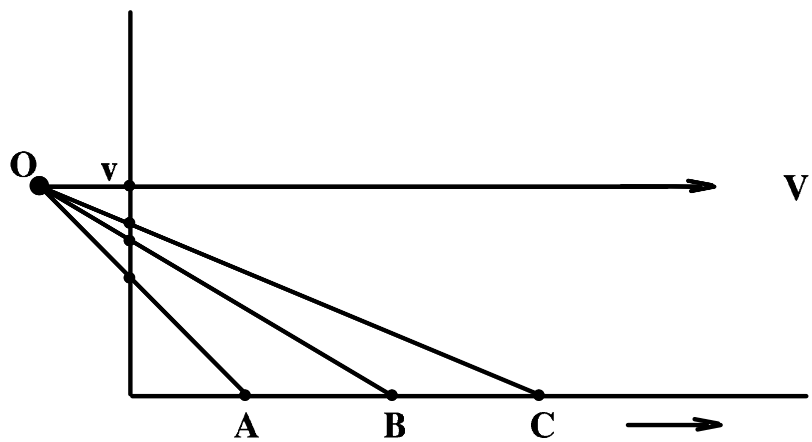

Figure 23.2
A one-dimensional construction of perspective viewing which illustrates the formation of a vanishing point.

Projective invariants – Perhaps the most significant contribution of projective geometry is the formulation of invariants under projective transformations. A wide variety of such invariants are available for sets of points and lines as well as higher order algebraic curves. The development and application of such invariants to computer vision is the focus of this book.

23.3 Geometry under perspective viewing

The initial understanding of the effects of perspective was developed in the context of artistic drawing. From the 15th century onwards, the problem of understanding and precisely constructing the effects of perspective viewing has been considered a key aspect of artistic drawing. The goal is to create a realistic impression of depth on a two-dimensional surface where the central phenomena which must be accounted for is the convergence of parallel lines at a vanishing point. A brief treatment of perspective construction will prove useful in establishing the effect of perspective on geometric properties.

A simple construction to illustrate the idea of the vanishing point is shown in Figure 23.2. The figure shows an image plane which is perpendicular to a ground plane and both planes are seen edge on. Under perspective viewing, an image point is constructed by the intersection of

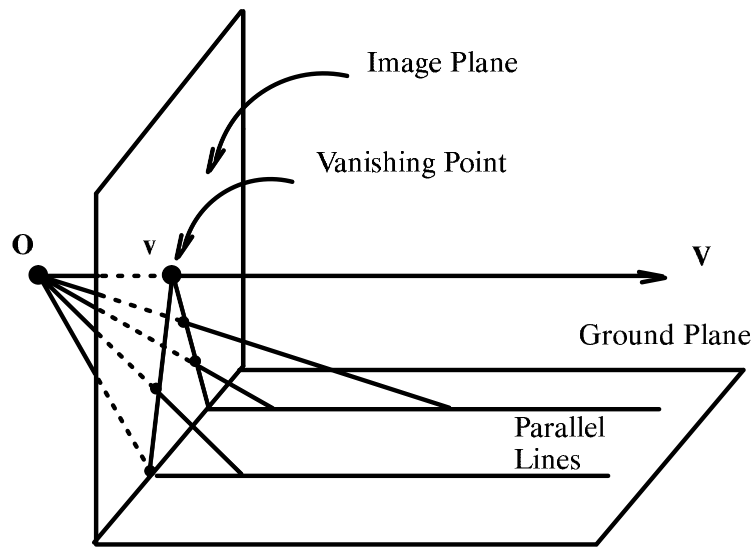

Figure 23.3
A two-dimensional construction of perspective viewing which illustrates the formation of a vanishing point.

a line from the eyepoint to the world point with the image plane. In this example all of the world points lie on the ground plane. Now consider a series of world points which are at increasing distances from the image plane such as A,B,C, ... V. The images of points far from O on the ground plane approach v and for point V, which is at infinity, the image is at the vanishing point, v. Note that the ray from the eyepoint through the vanishing point is parallel to the ground plane. It is also obvious from the figure that points equally spaced on the ground plane are not equally spaced in the image. This demonstrates that neither distance nor ratio of distances are preserved under perspective viewing.

In Figure 23.3 the convergence of parallel lines is illustrated. Here we see an oblique view of the ground plane and image plane and two parallel lines lie on the ground plane. As points on the two lines recede to infinity their corresponding image points converge to the *same* vanishing point. The vanishing point, $\mathbf{v}$, is the point at which the ray, $\overline{\mathbf{OV}}$, parallel to the two lines and passing through the eyepoint, intersects the image plane.

23.3.1 Perspective drawing

The fundamental property of perspective is that all image points are collinear with the eyepoint and their corresponding world point. As we have just seen, a vanishing point in the image plane defines a set of parallel lines in 3D world coordinates which are parallel to the ray

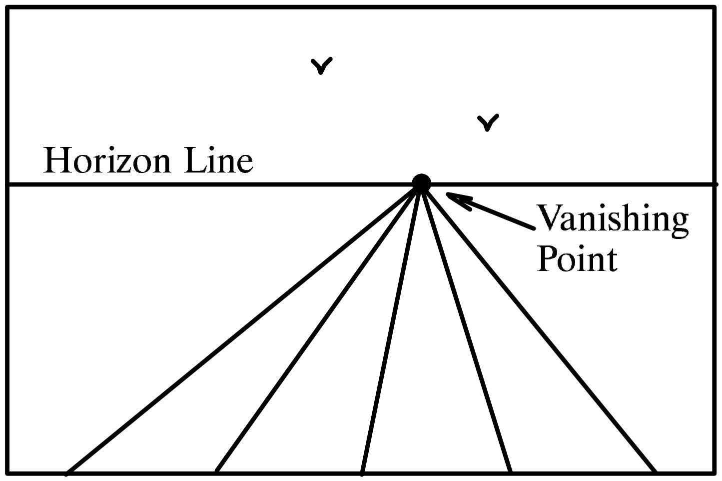

Figure 23.4
A perspective view of a set of parallel lines in the plane. All of the lines converge to a single vanishing point.

from the eyepoint, passing through the vanishing point. Conversely, the images of a set of lines parallel in space form a set of concurrent lines[5] which intersect at the vanishing point. This relationship between vanishing points and 3D line orientation provides the foundation for constructing realistic perspective views of the three-dimensional world.

First consider the case of one vanishing point and a set of coplanar parallel lines. A general perspective view of this configuration is shown in Figure 23.4. The drawing can be interpreted as a set of parallel lines, but the effect is not very convincing. Now consider two sets of parallel lines as on a tiled floor. The edges of the tiles will converge to two different vanishing points and the sketch shown in Figure 23.5 now begins to have a stronger impression of depth. The two vanishing points define a line in the image plane called the horizon line. Any set of parallel lines in the plane define a vanishing point which lies on the horizon line. The shape of general curved boundaries under perspective can be constructed by approximating the shape with a polygon. Each edge direction in the polygon has a corresponding vanishing point on the horizon line. All edges with a given direction must intersect at the same vanishing point. For example, the sides of a rectangle have two vanishing points and each edge must be inclined in the image so that

[5]Lines which all intersect at a common point. This configuration of lines is called a pencil.

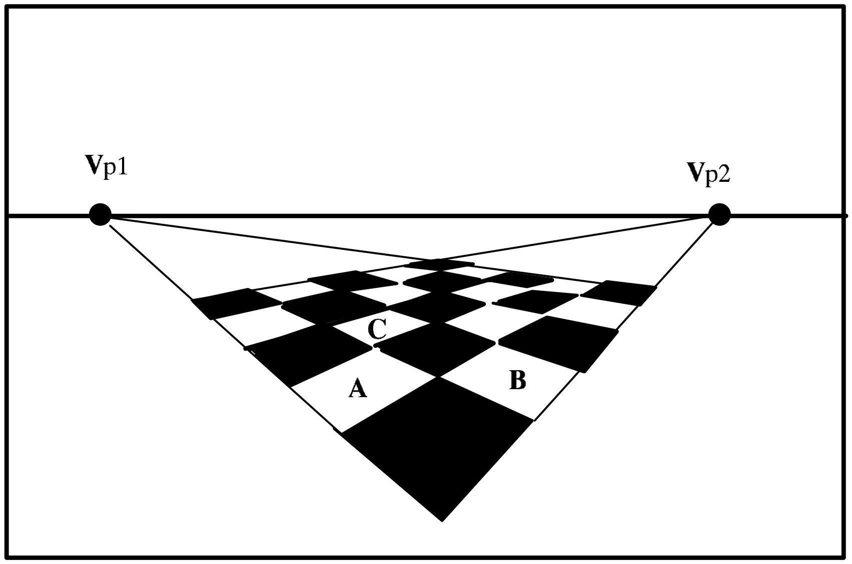

Figure 23.5
Different directions define different vanishing points. Here a tile floor is shown in perspective. The two edge directions of the tiles define two vanishing points on the horizon. The projected area of each tile is not equal in the image nor is the ratio between tile areas. For example the ratio of areas for B:A is about three times that of C:A.

parallel edges meet at the vanishing point when extended.

It has already been demonstrated that the ratio of distances on a line is not preserved under perspective. The tiled floor example demonstrates that ratio of areas of two figures is also not preserved. All tiles in the figure have the same area, but the ratio of the area between any two tiles in the perspective image can take on any value, depending on the choice of tiles. For example, the ratio of areas of the images of tiles A and B is about three times that of the area ratio of tile images C and A. This area ratio can increase without limit for tiles approaching the horizon.

An important case for vision applications is the circle. It is a common but incorrect notion that the center of a circle is preserved under a perspective transformation. The perspective view of a circle is constructed as shown in Figure 23.6. The figure shows a plan view of the circle and the construction lines required to construct a perspective view of the circle from the plan view. The constructions are based on the specification of vanishing points for the edges of a square circumscribed about the circle and the fact that line intersections are preserved by perspective. The construction clearly shows that the center of the perspective view of the circle, an ellipse, lies arbitrarily far from the projection of

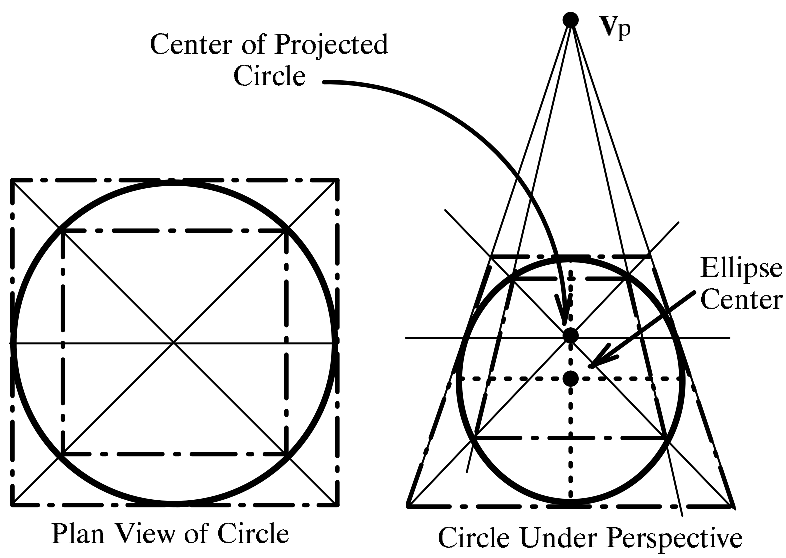

Figure 23.6
Perspective construction of a circle. Note that the centroid of the projection of the circle does not correspond to the image of the center of the circle.

the original center of the circle. The center of the circle always corresponds to the intersection of the diagonals of the circumscribed square. The ellipse must pass through the points of tangency of the edges of the circumscribed square and also (in this example) the major axis of the ellipse must correspond to the line joining the points of tangency of the square with the circle.

The perspective construction can be extended to the third dimension by adding more vanishing points. For the case of a cube there are three world plane orientations and from a general viewpoint, three vanishing points are required for the edges of the cube. In Figure 23.7, the structures define many vanishing points in the image. The vanishing points for three major orientations are indicated.

The concept of symmetry is also not meaningful under perspective. Consider the perspective drawing of the hourglass shape in Figure 23.8. The original shape has two axes of symmetry but under perspective viewing the image shape is not symmetrical[6].

[6]Symmetry is also lost under orthographic projection, but the concept of skewed symmetry can be defined which enables the recovery of symmetrical figures and constrains the pose of the figure planes [166].

Figure 23.7
A perspective drawing with major vanishing points indicated.

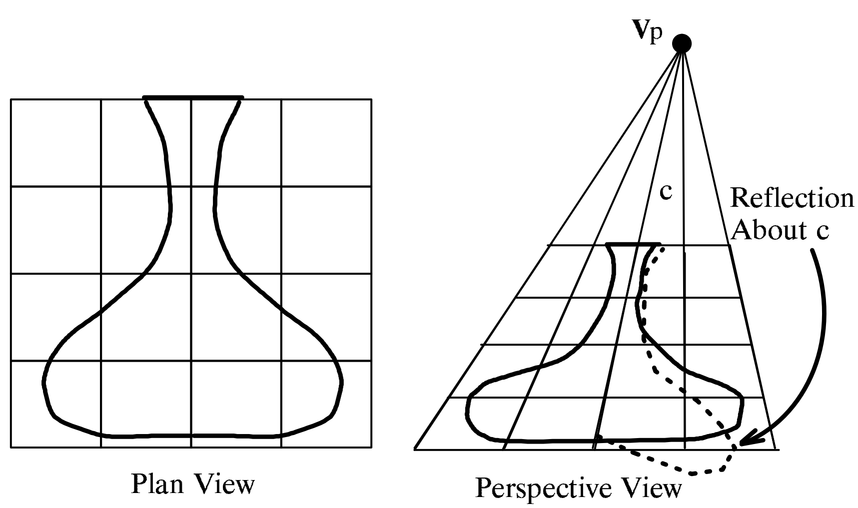

Figure 23.8
Symmetry is lost under perspective projection. The dashed line indicates a reflection of the perspectively transformed shape about the line c.

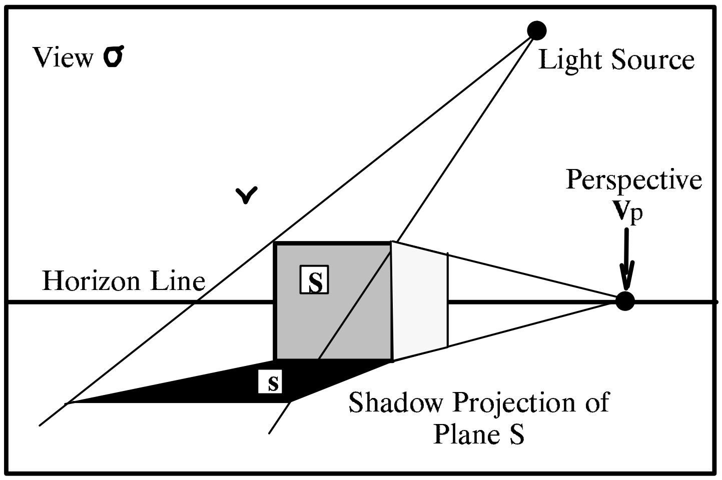

Figure 23.9
The perspective viewing of shadows requires the composition of two perspective projections, one from the light source to the ground plane, followed by a projection through the eyepoint to the image plane.

23.3.2 Pictures of pictures

So far we have considered the problem of constructing perspective views of two and three-dimensional geometric shapes. It is interesting to consider what happens if the scene to be projected itself contains a perspective image, such as the case of a room scene with a picture on the wall. Another important example of a chain of perspective views occurs in outdoor scenes where shadows are projected by a point light source. A simple shadow scene is illustrated in Figure 23.9 which shows the projection of the shadow of a shape onto the ground plane. The shape of the shadow is determined by the same collinear construction methods as in the construction of any perspective view. Any point on the boundary of the shadow is collinear with the corresponding point on the object boundary and the light source. A second perspective projection of the shadow is required to produce the image of the scene containing the shadow.

In general, there is no perspective geometry which will project the original shape, S, onto the image of its shadow, σ. That is, it is impossible to find any configuration of the center of projection and the image plane which will map $S \to \sigma$. The impossibility of representing any sequence of perspective projections by a single perspective view is easily demonstrated by perspective projection onto lines from points in

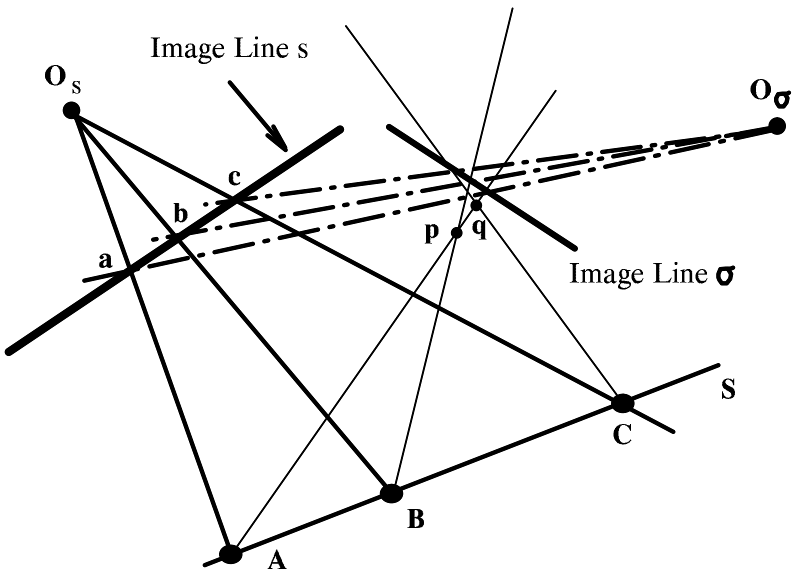

Figure 23.10
An example of the fact that a sequence of two or more perspective projections are not representable by a single perspective view. The figure shows that lines from $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ through the corresponding image points on σ intersect in distinct points, such as $\mathbf{p}$ and $\mathbf{q}$.

the plane. Figure 23.10 illustrates two 1D perspective projections. s is a projection of S and σ is a projection of s. Clearly, there is no common center of projection which is the intersection of all lines joining the corresponding points of σ and S.

The implication of this observation is that a more general mapping is required than perspective in order to explain all the geometric effects that can occur in perspective viewing. This more general mapping is called the *projective* transformation. Three key points about the projective transformation are as follows:

1. Any perspective projection can be represented by a projective transformation.

2. Any composite chain of perspective projections is not necessarily a perspective projection but is always a projective transformation.

3. Any projective transformation can be decomposed into two perspective transformations.

Consequently, it becomes necessary to develop a mathematical framework which characterizes the properties of the projective transformation. In the following we will review the theory of projective geometry which can account for the effects of perspective and projective image formation. In vision applications, we are most interested in the mapping from

3D space onto a 2D image plane. Unfortunately, most results in projective geometry have been developed for the projective plane which only provides the properties of the projection of a plane in space onto the image plane. However, in Section 23.10 we will use the results of projective geometry to model the perspective camera and provide a basis for the analysis required by vision applications.

23.4 The projective plane

23.4.1 The properties of the projective plane

The projective plane is a mathematical concept intended to model the geometric properties of a sequence of one or more perspective projections. In the projective plane model, transformations are represented by mappings of the plane onto itself. Thus a transformation can be viewed as a rearrangement of the points of the projective plane called a collineation. The behavior of geometric structures, such as lines, under collineations is the main focus of the theory of the projective plane.

The effect of a collineation on geometric properties differs from Euclidean transformations in two major aspects:

1. **Distance -** On the Euclidean plane the distance

$$\sqrt{(x-x_0)^2+(y-y_0)^2}$$

between two points, $\mathbf{P}$ and $\mathbf{P_0}$ is not affected by Euclidean transformations, i.e. translation and rotation. Under perspective, the distance between two points can be transformed to any value.

2. **Parallel Lines -** The image of parallel lines can be two intersecting lines when viewed under perspective.

The projective plane can be defined as a generalization of the Euclidean plane where some properties are removed. This generalization proceeds in two steps. First, the notion of distance is discarded, forming a structure called the *affine* plane. The main property which characterizes the affine plane is that parallelism is an invariant of affine transformations. Under affine transformations, the coordinates of points undergo anisotropic scaling, e.g., a square is transformed into an arbitrary parallelogram. The affine plane is often used in vision applications as a reasonable approximation to perspective image formation[7].

Second, the model removes the concept of parallel lines. All line pairs intersect in some unique point under perspective viewing. In order to account for the case where two lines are actually parallel, i.e. meet at infinity, the notion of an *ideal* point is introduced. As we shall see,

[7]The affine approximation is discussed in more detail in Section 23.10.

each parallel line orientation defines a different ideal point. The set of all these ideal points is a line added to the affine plane. For example, the horizon line in an image represents a line at infinity which has been projected into a line of finite points by the perspective transformation. In the projective plane model, ideal points are not distinguished from other points which leads to considerable simplification in analyzing perspective effects.

An affine plane with a line of ideal points adjoined and thereafter not distinguished is called the *projective plane.* The incidence axioms for the projective plane are as follows:

A1 Two distinct points determine a unique line.

A2 Two distinct lines determine a unique point.

Note that axiom A2 would not hold in the affine plane since parallel lines do not meet. The most important aspect of these axioms is that they are identical except for the words *point* and *line.* In fact, we could exchange the words and exchange A1 and A2. Thus, in the projective plane, lines and points are said to be *dual.* Any theorem (property) applying to lines also applies to points, and vice versa. In a statement involving both points and lines, the two words can be exchanged without affecting the truth of the statement. We will find this concept of duality very important in the application of projective geometry. Once a result has been worked out for points, a similar result for lines is obtained for free.

23.4.2 Models for the projective plane

As we have seen, the familiar Euclidean plane cannot be used to model the properties of projective transformations. New models are needed to provide intuition about the behavior of geometric relationships under projection. Perhaps the most useful model is provided by a set of rays in a three-dimensional space, R^3. As shown in Figure 23.11 all rays emanate from a common origin. Each ray represents a projective point. Only the direction of a ray is important in the model. Suppose an arbitrary plane, π, not passing through the origin, is constructed in R^3. The rays which intersect the plane correspond to points in the affine plane. Rays which are parallel to the plane model ideal points. The set of all rays parallel to π and passing through the origin is a one parameter family and is mathematically equivalent to a line, the ideal line. Since the plane is arbitrary, any point along a ray is equivalent to any other point. Also, there is no real distinction among the rays as to affine or ideal points, since π is arbitrary.

The plane through the origin defined by any two distinct rays is a model for the projective line. To see this, consider each of the two incidence axioms, A1 and A2. First, two rays always define a unique

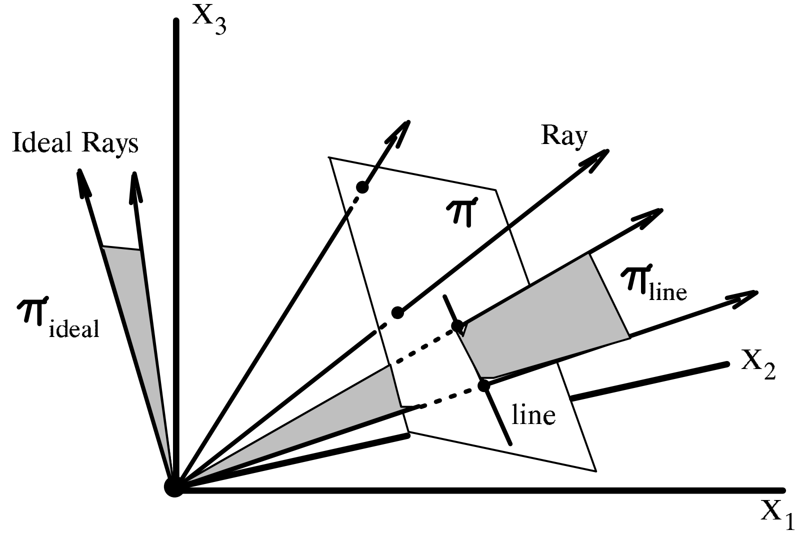

Figure 23.11
A model for the projective plane can be constructed by rays in 3D space. The rays correspond to points in the projective plane. Two rays through the origin define a unique plane though the origin. Any plane through the origin corresponds to a projective line.

plane through the origin, which satisfies A1. Second, a unique ray always exists which is the intersection of two such planes, thus providing a model for axiom A2. The affine plane, π, intersects the plane defined by two rays, π_{line}, in an affine line. If two lines are parallel in the affine plane, they meet on one of the rays which is parallel to π. The set of all rays parallel to π form a plane, π_{ideal}. Again, this plane is just a model for a projective line and is not distinguished from other planes.

The ray model also illustrates the behavior of a perspective mapping between planes. In this case, two planes are introduced, Π and π. The origin of ray space can be considered to be a center of projection and a ray intersects both planes in the corresponding perspectively transformed points. The same relationship can be achieved by considering π and Π to be a single plane which is transformed from one plane to the other by transforming the coordinate system of ray space. The ray intersections with π and Π are a model for the projective transformation of points. From this model for a perspective transformation, it is easy to see that ideal points can be mapped into finite points and vice versa.

It is possible to model the arbitrary relationship between π and Π by rotating and scaling the rays in R^3. All positions and orientations of π with respect to the rays can be achieved by a rotation and scaling of R^3. The position along a ray of its intersections with π is not important so the transformation of R^3 can involve both rotation and anisotropic scaling in general. Rotation and scaling of R^3 is represented by a general 3×3 matrix multiplication, as discussed below. It follows that any composition of perspective projections also corresponds to some combined rotation and scaling of ray space.

This ray model represents all of the geometric properties of the projective plane by interpreting rays as *points* and planes through the origin as *lines*. An analytical theory can be established by introducing the coordinates of the three-dimensional ray space.

23.4.3 Homogeneous coordinates

This analytic geometry of the projective plane is a direct consequence of the algebraic properties of the coordinates of the three-dimensional ray space. According to the model just developed, a point in the projective plane is represented by three Cartesian coordinates[8], $\mathbf{p} = (x_1, x_2, x_3)^t$, which represents a ray through the origin in three-dimensional space. $(x_1, x_2, x_3)^t$ are called homogeneous coordinates because algebraic expressions representing forms, such as conics, become homogeneous equations[9] when expressed as polynomials in $(x_1, x_2, x_3)^t$. Only the direction of the ray is important, so all points of the form $\lambda\mathbf{p} = (\lambda x_1, \lambda x_2, \lambda x_3)^t$ are equivalent. Conversely, all projective properties of a point must hold regardless of the value of λ. Clearly, the direction of a ray of zero length is not defined. The corresponding homogeneous coordinates $(0, 0, 0)$ has no meaning, and is undefined in the projective plane.

A relationship to conventional Cartesian coordinates in the plane, (x, y), can be established by constructing a special plane, π_e, which is perpendicular to the x_3-axis and at unit distance along x_3. The intersection of the ray $\mathbf{p}$ is the point, $\mathbf{p}_e = (x, y, 1)$, where the pair (x, y) corresponds to the standard Cartesian coordinates of $\mathbf{p}$. As we discussed above, a ray parallel to π_e is called an ideal point. Any ideal point therefore has $x_3 = 0$. The condition $x_3 = 0$ defines a line called the ideal line.

It is not necessary to specify a unit distance along x_3 for the location of π_e. Instead the Cartesian coordinates corresponding to a projective point are defined by,

$$\mathbf{p}_e = (\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1)^t = (x, y, 1)^t$$

so that the position of the plane does not affect the value of the Cartesian

[8]In the following development, coordinate vectors are considered to be column vectors.

[9]Recall that a homogeneous polynomial equation has all monomial terms of equal total degree. As a consequence, if q is a polynomial in the coordinates, then $q(\lambda x_1, \lambda x_2, \lambda x_3) = \lambda^k q(x_1, x_2, x_3)$, where k is the degree of each monomial.

coordinates. It is not essential that the plane be perpendicular to x_3 but this is the standard convention so that ideal points are denoted by $x_3 = 0$.

23.4.4 The projective line

The coordinate representation of a line in the projective plane is derived from the analytic representation of a general plane through the origin of ray space. The equation of this plane is given by,

$$u_1 x_1 + u_2 x_2 + u_3 x_3 = 0 \tag{23.1}$$

the plane coefficients $\mathbf{u} = (u_1, u_2, u_3)^t$ correspond to the homogeneous coordinates for the projective line. Again, $\lambda\mathbf{u}$ is the same line as $\mathbf{u}$. Note that the equation is homogeneous, since the degree of each term is the same. The case of $u_3 = 0$ corresponds to a line through the origin. The ideal line is $\mathbf{u} = (0, 0, 1)^t$ which has the equation, $x_3 = 0$. The projective equation of a line can be represented in various vector and vector array notations.

$$\mathbf{u} \cdot \mathbf{p} = \mathbf{u}^t\mathbf{p} = \mathbf{p}^t\mathbf{u} = 0.$$

The duality of points and lines is indicated by the symmetric form of these equations. That is, the role of $\mathbf{u}$ and $\mathbf{p}$ can be interchanged without affecting the form of the equation. The homogeneous projective form of the line can be related to the standard Cartesian line equation. In Cartesian coordinates, the equation of a line is:

$$n_x x + n_y y - d = 0$$

where $\mathbf{n} = (n_x, n_y)^t$ is the normal to the line and d is the distance from the origin to the line in the direction perpendicular to the line. We can compare this expression to the homogeneous line equation (23.1) and determine the relationship between Cartesian line parameters and homogeneous line coefficients. The line normal components are,

$$n_x = -d\frac{u_1}{u_3} \qquad n_y = -d\frac{u_2}{u_3} \tag{23.2}$$

showing that the normal to the Cartesian line is just the projection onto the xy plane of the normal, (u_1, u_2, u_3), of the corresponding plane in ray space.

The concept of the vanishing point discussed earlier can be related to homogeneous coordinates as follows. Any vanishing point, $\mathbf{p}^v$, corresponds to an ideal point with $x_3 = 0$. So $\mathbf{p}^v$ must be of the form, $\mathbf{p}^v = (p_1^v, p_2^v, 0)$. The equation of a line incident with the vanishing point is given by,

$$u_1 p_1^v + u_2 p_2^v = 0.$$

So, for all such lines

$$\frac{u_1}{u_2} = -\frac{p_2^v}{p_1^v}$$

and from equation (23.2) above,

$$\frac{n_x}{n_y} = -\frac{p_2^v}{p_1^v}$$

so (p_1^v, p_2^v) must correspond to the direction of the line, i.e.

$$(n_x, n_y) \cdot (p_1^v, p_2^v) = n_x p_1^v + n_y p_2^v = 0.$$

It follows that any line (in the plane) incident with this vanishing point will have the same direction, since the ratio n_y/n_x is fixed. When this set of parallel lines is projectively mapped, the lines are no longer necessarily parallel. However they will still all be incident with the projection of $\mathbf{p}^v$. This condition shows that the concept of parallelism is not meaningful in projective geometry since there is nothing to identify $\mathbf{p}^v$ as an ideal point. In order to define the concept of parallelism, it is necessary to distinguish a set of ideal points from the other points on the projective plane. Lines are parallel if they intersect at an ideal point. Augmenting the projective plane with the additional structure of a line of ideal points results in an affine plane.

It is necessary to specify a line of ideal points in the plane before parallelism can be defined.

23.4.5 Projective transformations

As we discussed above, a projective transformation between two projective planes can be represented by a general linear transformation of ray space, $(x_1, x_2, x_3) = \mathtt{T}(X_1, X_2, X_3)^t$. It can be shown that all properties of a general projective transformation are accounted for by this matrix transformation. Conversely, any (non-singular) linear transformation of homogeneous coordinates defines a projective transformation of the projective plane. This form for the projective transformation is a key benefit of introducing homogeneous coordinates, since many important results can be obtained directly by manipulating simple matrix and vector expressions.

Since the projective plane has three homogeneous coordinates the transformation is represented by a 3×3 matrix with 8 essential parameters. The overall scale of the matrix is not important since all projective points are equivalent up to the multiplier, λ. The set of distinct projective transformations is an eight-dimensional subspace of the nine-dimensional space defined by the matrix elements.

The general projective transformation from one projective plane, Π, to another, π, is represented as[10]

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

or

$$\mathbf{x} = \mathtt{T}\mathbf{X}.$$

If the transformation is represented in Cartesian coordinates the non-linear nature of the projective transformation in Euclidean or affine space is apparent.

$$\begin{aligned} x &= \frac{x_1}{x_3} = \frac{t_{11}X + t_{12}Y + t_{13}}{t_{31}X + t_{32}Y + t_{33}} \\ y &= \frac{x_2}{x_3} = \frac{t_{21}X + t_{22}Y + t_{23}}{t_{31}X + t_{32}Y + t_{33}}. \end{aligned} \tag{23.3}$$

Again, the overall scale factor of $\mathtt{T}$ does not affect the Cartesian coordinates since the same factor appears in the numerator and denominator of each expression. This linear rational form represents the effect on Cartesian coordinates of a projective transformation and accounts for a *sequence* of central projections between two planes in space. Consequently, it is of major importance in vision applications.

23.4.6 Projective transformation of lines

Since points and lines are dual in the projective plane, the transformation of line coordinates is also a linear transformation. Consider the equation for a point incident with a line defined earlier,

$$\begin{aligned} U_1X_1 + U_2X_2 + U_3X_3 &= 0 \\ \mathbf{U}^t\mathbf{P} &= 0. \end{aligned}$$

If a point, $\mathbf{P}$, transforms as, $\mathbf{p} = \mathtt{T}\mathbf{P}$, then $\mathbf{P} = \mathtt{T}^{-1}\mathbf{p}$. Substituting this inverse transformation into the line equation yields

$$\mathbf{U}^t\mathtt{T}^{-1}\mathbf{p} = 0.$$

It is seen that the collinearity of points is preserved under a projective transformation since the general form of the dot product is not affected by the linear homogeneous transformation. Further, the transformed line equation is $\mathbf{u}^t\mathbf{p}$, and the transformed line coordinates, $\mathbf{u}$, must be

$$\mathbf{u} = [\mathtt{T}^{-1}]^t\mathbf{U}$$

[10]In the following, capital letters are used to indicate the source objects, and corresponding small letters are used to represent the destination transformed objects.

which is often represented more compactly as

$$\mathbf{u} = \mathtt{T}^{-t}\mathbf{U}.$$

Thus, lines in the projective plane transform linearly, just as points, but the corresponding transformation matrix is the transpose of the inverse of the matrix defining the point transformation.

This mapping of points into points and lines into lines is called a collineation. The term collineation implies that any projective transformation of the plane preserves the collinearity of a set of points. Since the form of the transformation for points and lines is the same, it is natural to also consider the possibility of a transformation of the projective plane which takes points into lines and lines into points. Such a transformation is called a correlation. We will find the concept of the correlation useful in interpreting the projective properties of algebraic curves. For example, a conic defines a correlation between poles and corresponding polar lines[11].

23.4.7 Four points define a projective transformation

The projective transformation matrix, $\mathtt{T}$, requires eight independent parameters to define a unique mapping. Since each point in the plane provides two Cartesian coordinate equations, it is necessary to find four point correspondences between two projectively transformed planes to define the transformation matrix uniquely. The overall scale of $\mathbf{T}$ is arbitrary, so we can choose $t_{33} = 1$. Let the four corresponding points be represented by, $(\lambda_i x_i, \lambda_i y_i, \lambda_i)^t = \mathtt{T}(X_i, Y_i, 1)^t$. The resulting linear system of equations is,

$$\begin{bmatrix} X_1 & Y_1 & 1 & 0 & 0 & 0 & -x_1X_1 & -x_1Y_1 \\ 0 & 0 & 0 & X_1 & Y_1 & 1 & -y_1X_1 & -y_1Y_1 \\ X_2 & Y_2 & 1 & 0 & 0 & 0 & -x_2X_2 & -x_2Y_2 \\ 0 & 0 & 0 & X_2 & Y_2 & 1 & -y_2X_2 & -y_2Y_2 \\ X_3 & Y_3 & 1 & 0 & 0 & 0 & -x_3X_3 & -x_3Y_3 \\ 0 & 0 & 0 & X_3 & Y_3 & 1 & -y_3X_3 & -y_3Y_3 \\ X_4 & Y_4 & 1 & 0 & 0 & 0 & -x_4X_4 & -x_4Y_4 \\ 0 & 0 & 0 & X_4 & Y_4 & 1 & -y_4X_4 & -y_4Y_4 \end{bmatrix} \begin{bmatrix} t_{11} \\ t_{12} \\ t_{13} \\ t_{21} \\ t_{22} \\ t_{23} \\ t_{31} \\ t_{32} \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{bmatrix}. \quad (23.4)$$

The existence of this linear system ensures, in principle, the uniqueness of $\mathbf{T}$, given four point correspondences, provided that no three of the points are collinear[12].

This result leads immediately to a canonical projective coordinate system, based on four points, where the properties of geometric figures can

[11] See Section 23.7.1.

[12] When more than four points are available, singular value decomposition methods can be used to produce least squares solution. Also, by using a singular value decomposition approach, it is not necessary to single out a particular element of $\mathtt{T}$, e.g. t_{33}.

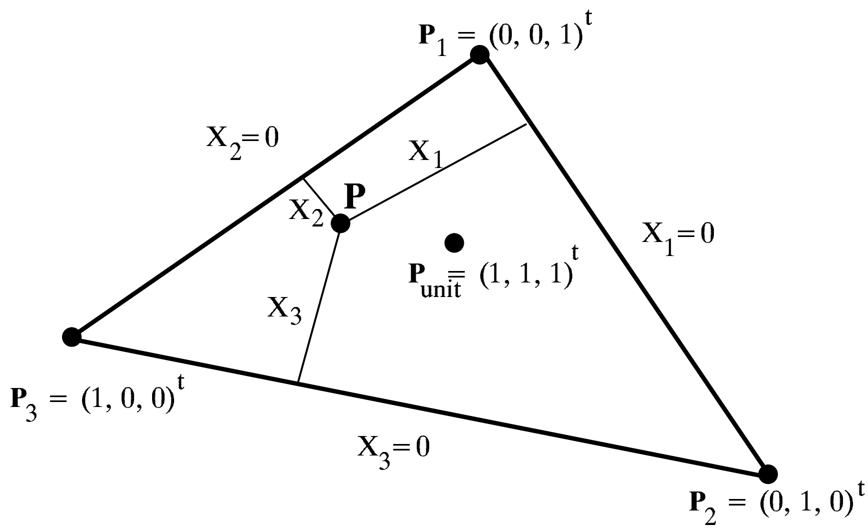

Figure 23.12
The standard projective coordinate frame defined by four points, and called the triangle of reference. Homogeneous coordinates are defined as the perpendicular distance from a point to the edges of the triangle. The distance scale is set by the unit point.

be invariantly represented. An obvious choice is to select the canonical frame to be a unit square where the transformed points of reference have coordinates, $\{(0,0,1)^t, (1,0,1)^t, (0,1,1)^t, (1,1,1)^t\}$. Any reference quadrilateral can be projectively transformed onto this unit square and then the geometric relationships of other points and lines can be invariantly represented in the canonical frame. This approach is taken in the representation of planar curved shapes to provide an invariant signature of the curve for recognition[13].

Another canonical coordinate frame is introduced in classical projective geometry which simplifies the analysis of some geometric configurations and in particular is useful in the analysis of the projective properties of conics. The coordinate system, based on a triangle of reference, is shown in Figure 23.12. Three of the points are used to define a triangle of reference. Standard coordinates are assigned to the points as shown. The fourth point is assigned coordinates, $\mathbf{p}_{unit} = (1,1,1)^t$ and is called the unit point. Once four points have been specified, the homogeneous coordinates of the Cartesian point representation, $\mathbf{p} = (x,y)^t$ are defined by,

$$x_i = \lambda(u_{i1}x + u_{i2}y + u_{i3}) \qquad (23.5)$$

[13]See Chapter 11.

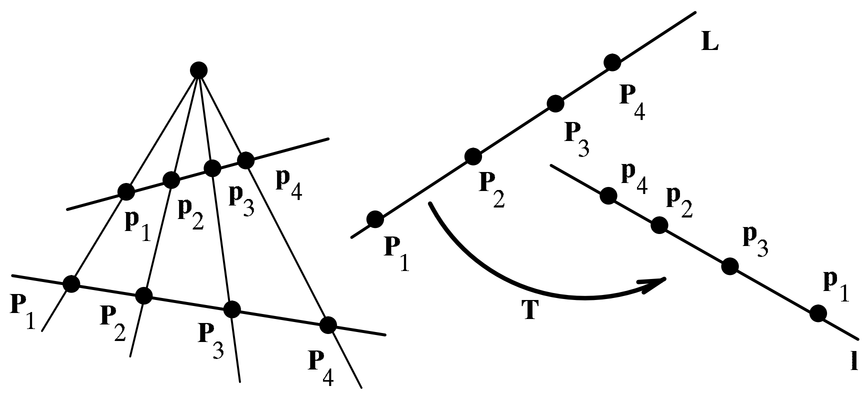

Figure 23.13
The cross-ratio for all lines cutting the pencil on the left is the same. This configuration corresponds to perspective projection onto a line. The figure to the right illustrates a general projective transformation on the line.

where $\mathbf{u_i}$ are the line coordinates of the edges of the triangle of reference. Equation 23.5 represents the line equation for a point lying on one of the sides of the triangle. The value of this expression is zero if the point $(x, y, 1)^t$ lies on the line. Otherwise the value is proportional to the perpendicular distance from the point to the line. The value of each homogeneous coordinate x_i is the Euclidean distance of the given point from the corresponding edge of the triangle of reference. For example x_1 is the distance to the line, $\overline{P_1P_2}$, i.e. $x_1 = 0$. The arbitrary distance scale factor λ is determined by letting $\mathbf{p} = \mathbf{p}_{unit}$.

23.5 The cross-ratio

23.5.1 The definition of the cross-ratio

Perhaps the most important result for the theme of this book is the fact that the cross-ratio of four points on a line is preserved under projective transformations. There are many results in projective geometry which result in an interpretation in terms of the cross-ratio. It seems likely that all invariant properties of a geometric configuration can ultimately be interpreted in terms of some number of cross-ratio constructions.

The cross-ratio is defined with respect to Figure 23.13. As we discussed earlier in Section 23.3, the ratio of distances is not preserved under a projective transformation, however, the ratio of ratios of distances is invariant. The cross-ratio is defined by

$$Cr(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4) = \frac{(X^3 - X^1)(X^4 - X^2)}{(X^3 - X^2)(X^4 - X^1)} \tag{23.6}$$

where $\{X^1, X^2, X^3, X^4\}$ represent the corresponding positions of each point along the line, e.g. $(X^3 - X^1)$ is the distance between points $\mathbf{P}_3$ and $\mathbf{P}_1$.

23.5.2 The invariance of the cross-ratio

In order to show the invariance of the cross-ratio it is first necessary to show the effect of projective transformations on the coordinates of points on a line. Any point on a line can be represented in terms of only two homogeneous coordinates[14]. and points on this line are represented by $\mathbf{P} = (X_1, X_2)^t$, where X_1, X_2 are homogeneous line coordinates. The cartesian position of a point on the line is given by $X = X_1/X_2$. Similarly, the model for a projective line is provided by a set of rays through the origin of a plane, R^2. Cartesian points are the intersection of the rays with an arbitrary line $\mathbf{l}$ in this 2D space. Again, the projective mapping between lines is represented by linear transformations of ray space.

The projective mapping between lines reduces to a 2×2 homogeneous transformation matrix, $\mathtt{T}$ so that $\mathbf{x} = \mathtt{T}\mathbf{X}$ where $(x_1, x_2)^t$ are the homogeneous coordinates on the transformed line. In this case, there are three essential parameters needed to define $\mathtt{T}$ since again the overall scale of $\mathtt{T}$ is not important. The cartesian position of a point on the line is given by $x = x_1/x_2$. It follows that projective transformations on the line are of the form,

$$x = \frac{t_{11}X + t_{12}}{t_{21}X + t_{22}}$$

which is the 2D homogeneous form of equation (23.3). Now consider the determinant of the 2×2 matrix which is formed from two points on a line, $\mathbf{P}_1, \mathbf{P}_2$,

$$D(12) = |\mathbf{P}_1 \mathbf{P}_2|$$

which expands to,

$$D(12) = \begin{vmatrix} X_1^1 & X_1^2 \\ X_2^1 & X_2^2 \end{vmatrix}.$$

[14]In general the number of homogeneous coordinates is $dim(S) + 1$, where $dim(S)$ is the dimension of the geometric space.

Carrying out the determinant,

$$\begin{aligned} D(12) &= X_1^1 X_2^2 - X_1^2 X_2^1 \\ &= X_2^1 X_2^2 \left(\frac{X_1^1}{X_2^1} - \frac{X_1^2}{X_2^2}\right) \\ &= X_2^1 X_2^2 (X^1 - X^2). \end{aligned} \tag{23.7}$$

The constant of proportionality is just the arbitrary projective scale factor for each point. We can make the idea of the scale factor more explicit,

$$\mathbf{P}_1 = \Lambda_1 (X^1, 1)^t$$
$$\mathbf{P}_2 = \Lambda_2 (X^2, 1)^t.$$

Here, Λ_1, Λ_2 are the point scale factors. The result of equation (23.7) becomes, $D(12) = \Lambda_1 \Lambda_2 (X^1 - X^2)$. $D(12)$ is transformed under projection to $d(12) = |\mathtt{T}\,[\mathbf{P}_1 \mathbf{P}_2]|$, so

$$d(12) \;=\; \lambda_1 \lambda_2 (x^1 - x^2) = \Lambda_1 \Lambda_2 (X^1 - X^2) |\mathtt{T}|$$

where λ_1, λ_2 are just the arbitrary scale factors on the transformed line. Next, consider the ratio of determinants of point pairs,

$$R = \frac{D(31)}{D(32)} = \frac{|\mathbf{P}_3 \mathbf{P}_1|}{|\mathbf{P}_3 \mathbf{P}_2|} = \frac{\Lambda_1 (X^3 - X^1)}{\Lambda_2 (X^3 - X^2)}.$$

The transformed ratio, r, is given by

$$r = \frac{|[\mathbf{p}_3 \mathbf{p}_1]|}{|[\mathbf{p}_3 \mathbf{p}_2]|} = \frac{\lambda_1 (x^3 - x^1)}{\lambda_2 (x^3 - x^2)} = \frac{\Lambda_1 (X^3 - X^1)}{\Lambda_2 (X^3 - X^2)}.$$

The determinant of the transformation matrix is eliminated from the ratio. However, in order to eliminate the effect of the projective scale factors it is necessary to take the ratio of determinants from four point pairs. It is seen that any combination of ratios which has a point appearing the same number of times in the numerator as the denominator will eliminate $|\mathtt{T}|$ and the scale factors λ_i and Λ_i. For example, we can define,

$$Cr(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4) = \frac{D(31) D(42)}{D(32) D(41)}$$

and it easily follows that $Cr(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = Cr(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4)$ since both $|\mathtt{T}|$ and the projective scale factors, λ_i and Λ_i, cancel.

23.5.3 Order on the projective line

The development of the invariance of the cross-ratio suggests that other permutations of the points in the definition of the ratio will also lead to a scalar invariant. The four points can be permuted 4!(or 24) different ways. There are only six distinct values of the cross-ratio within the 24 permutations. If the cross-ratio for the standard definition of equation (23.6) is defined as τ then the six distinct values are related by the set,

$$\{\tau, \frac{1}{\tau}, 1-\tau, \frac{1}{1-\tau}, \frac{\tau-1}{\tau}, \frac{\tau}{\tau-1}\}.$$

The existence of these six different values is somewhat annoying in using the cross-ratio as an index for recognition since the order of points along a line can be permuted by a projective transformation. The possibility of permutation prevents a direct correspondence between points across views and therefore all six values of the cross-ratio must be considered as an index[15]. It is reasonable to expect the order of points to be reversed, as when we look at an object boundary from behind. In this case the value of the cross ratio is unchanged. Also, since a point at infinity can be transformed to a finite vanishing point, the point order is cyclically permuted as one of the points recedes to infinity and emerges on the other end of the line as in the permutation $\{1,2,3,4\} \rightarrow \{2,3,4,1\}$.

23.5.4 The cross-ratio of lines

Since points and lines are dual, there exists an equivalent cross-ratio for lines. The dual relation to collinearity is incidence at a point. A cross-ratio is defined on four lines which are incident at a single point. Any set of lines incident at a common point is called a pencil. Since the lines all share a single point the set can be described by a single parameter which defines the orientation of the lines. Note the dual notion of points along a line where a single parameter defines the point position on the line.

The derivation proceeds as before if we take the common point of intersection to lie at the origin. Then the line coordinates are of the form $\mathbf{U} = (U_1, U_2, 0)$ and the ratio $u = -\frac{U_1}{U_2}$ corresponds to the gradient of a line. This parameter u sweeps out the lines of the pencil.

The cross-ratio of the pencil can be defined in terms of the angles between the lines as shown in Figure 23.14. and is given by

$$Cr(\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3, \mathbf{U}_4) = \frac{\sin\alpha_{13}\sin\alpha_{24}}{\sin\alpha_{23}\sin\alpha_{14}}.$$

[15] It is shown in Chapter 5 that a rational function of the cross-ratio value can be defined which is independent of the effects of permutation. This is the j-invariant defined by

$$j(\tau) = \frac{(\tau^2 - \tau + 1)^3}{\tau^2(\tau-1)^2}.$$

So for example, $j(\tau) = j(1-\tau)$.

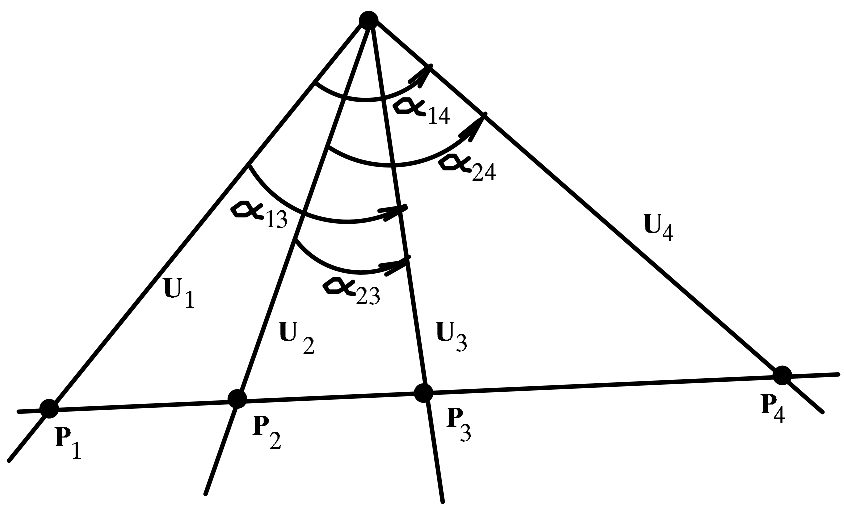

Figure 23.14
The dual configuration for the cross-ratio. The pencil of lines has a cross-ratio defined by the angle between lines. By Pappus' theorem, any line intersecting the pencil has the same cross-ratio for the points of intersection of the line with the pencil.

Now for any line which cuts the pencil, the four points of intersection of the line and the pencil define a cross-ratio on the line. Pappus' theorem states that if the points of intersection are $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4\}$, then $Cr(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4) = Cr(\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3, \mathbf{U}_4)$. For a proof of this see Springer [270].

23.5.5 The cross-ratio and projective coordinates

An important use of the cross-ratio in computer vision is the idea of *transfer*. It is generally the case that invariants can be used to establish the position of transformed points once the location of a few reference features are available in the target coordinate frame. The cross-ratio on the line provides a simple example of transfer.

Suppose we have three point correspondences between two projective transformations of a line. The position of any fourth point can be derived, given the cross-ratio of the point with respect to the first three points. Given the four points, $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{X}\}$, the cross-ratio, $Cr(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{X}) = Cr(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{x}) = \tau$ is an invariant. With the correspondences $\{\mathbf{p}_i, \mathbf{P}_i\}$ $i \in \{1, .., 3\}$, the location of any fourth point on the line, $\mathbf{x}$, may be determined from the cross-ratio as

$$x = \frac{x^2(x^3 - x^1) - \tau x^1(x^3 - x^2)}{(x^3 - x^1) - \tau(x^3 - x^2)}. \tag{23.8}$$

This result is obtained by solving the equation[16], $Cr(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{x}) = \tau$ for x. Note that the position of x in the transformed frame is computed without determining the elements of the transform matrix. Equivalently, the positions of the reference points could be used to compute the three essential parameters of the transformation matrix. We will extend this concept to multiple views of 3D point sets in Section 23.11.

23.6 Conics

23.6.1 The conic is defined by the cross-ratio

The properties of conics in the projective plane have proved very useful in vision applications. Perhaps the most dramatic example of the projective nature of the conic is that it is a curve which can be constructed from the cross-ratio. This construction for the conic is analogous to the circle in Euclidean geometry where the circle is a locus of points with constant Euclidean distance from a given point and distance is an invariant under Euclidean transformations. The construction is a result of Chasles' theorem,

Given four points in the plane, no three collinear. Construct a pencil of lines from another point in the plane and the four given points. The locus of such points that form a pencil with a fixed cross-ratio is a conic curve.

The conic is a curve defined directly in terms of a projective invariant property and therefore it is not surprising that it plays a central role in projective geometry.

23.6.2 The quadratic form of the conic

Most of the analytic results for the conic are derived from the quadratic homogeneous expression defining a conic curve in the plane,

$$AX_1{}^2 + BX_1X_2 + CX_2{}^2 + DX_1X_3 + EX_2X_3 + FX_3{}^2 = 0$$

Given that a point in the projective plane is represented in homogeneous coordinates, $\mathbf{X} = (X_1, X_2, X_3)^t$, the quadratic conic form can be represented as a matrix expression,

$$\mathbf{X}^t \mathtt{C} \mathbf{X} = 0$$

The conic coefficient matrix is given by

$$\mathtt{C} = \begin{bmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{bmatrix}.$$

Note that the conic coefficient matrix is symmetric.

[16] Recall that the Cartesian positions, x^i, are given by $x^i = x_1^i / x_2^i$.

Projective transformation of the conic – First define the transformation of a point, $\mathbf{X}$ as $\mathbf{x} = \mathtt{T}\mathbf{X}$. Equivalently, $\mathbf{X} = \mathtt{T}^{-1}\mathbf{x}$. Substituting into the conic form,

$$\mathbf{X}^t\mathtt{C}\mathbf{X} = \mathbf{x}^t[\mathtt{T}^{-1}]^t\mathtt{C}\mathtt{T}^{-1}\mathbf{x} = \mathbf{x}^t\mathtt{T}^{-t}\mathtt{C}\mathtt{T}^{-1}\mathbf{x}$$

This is a quadratic form $\mathbf{x}^t\mathtt{c}\mathbf{x}$ where

$$\mathtt{c} = \mathtt{T}^{-t}\mathtt{C}\mathtt{T}^{-1},$$

and, again, is a symmetric matrix:

$$\mathtt{c}^t = \mathtt{T}^{-t}\mathtt{C}^t\mathtt{T}^{-1} = \mathtt{T}^{-t}\mathtt{C}\mathtt{T}^{-1} = \mathtt{c}$$

which represents a conic. Thus, a conic is transformed to a conic under projection.

23.6.3 Equivalence of conics under projection

If we define $\mathtt{S} = \mathtt{T}^{-1}$ then the transformed conic and the original conic are related by

$$\mathtt{c} = \mathtt{S}^t\mathtt{C}\mathtt{S}.$$

This matrix relationship defines the *congruence* of two matrices, $\mathtt{c}$ and $\mathtt{C}$[17]. It is a standard theorem of matrix algebra that every real symmetric matrix is congruent to a diagonal matrix, so we can write

$$\mathtt{c} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

Thus we can always transform a conic to be centered on the origin and with its principal axes aligned with the Cartesian coordinate frame. In this coordinate frame, the conic equation is

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = 0.$$

Now in order to have a conic with real points it follows that one of the λ_i must be negative. So without loss of generality, we can assume the following form[18] for $\mathtt{c}$,

$$\mathtt{c} = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & -\gamma^2 \end{bmatrix}.$$

[17]The notion of the congruence of two matrices may not be familiar. The concept should not be confused with the congruence of geometric figures.

[18]Note that this form is general, in that a matrix with two negative elements can be returned to this form by scaling by -1.

This configuration is general since the negative coefficient can always be permuted to any position along the diagonal by a congruence operation. Next we apply the following transformation,

$$\mathtt{S} = \begin{bmatrix} 1/\alpha & 0 & 0 \\ 0 & 1/\beta & 0 \\ 0 & 0 & 1/\gamma \end{bmatrix}.$$

The resulting conic coefficient matrix is

$$\mathtt{c} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Note that these congruencies can be combined into a single congruence operation by multiplying the individual S matrices. The combined S matrix corresponds to a single projective transformation.

The final conic equation, $x_1^2 + x_2^2 - 1 = 0$, represents a circle of unit radius centered on the origin. Thus any conic can be projectively transformed into a circle. In fact, any two conics are projectively equivalent. Since any conic can be transformed into a circle, an inverse transform can be applied to map the circle back to any other arbitrary conic.

23.6.4 The line conic

A conic is a self-dual figure. That is, it can be considered as a locus of points or as an envelope of tangent lines as shown in Figure 23.15. The latter view is referred to as a line conic. The key point is that the equation describing the line coefficients of the tangents is also a quadratic form. Let $\mathbf{X}^t\mathtt{C}\mathbf{X} = 0$ define a point conic, then the corresponding line conic equation is $\mathbf{U}^t\mathtt{L}\mathbf{U}$, where $\mathtt{L} = |\mathtt{C}|\mathtt{C}^{-1}$. The interpretation is that any line, $\mathbf{U}$, which is tangent to the conic satisfies the line conic equation. The line conic transforms as, $\mathtt{l} = \mathtt{TLT}^t$, where $\mathtt{T}$ is the point transformation matrix.

23.7 Projective properties of the conic

23.7.1 The polar of a conic

Given a point, $\mathbf{P}$, in the plane, construct the tangents from $\mathbf{P}$ to a conic $\mathtt{C}$ as illustrated in Figure 23.15. There are two tangents to the conic from $\mathbf{P}$. The two points of tangency on the conic define a line, $\mathbf{U}$, which is called the polar of point $\mathbf{P}$ with respect to the conic $\mathtt{C}$. Conversely, $\mathbf{P}$ is called the pole of line $\mathbf{U}$.

The line representation of the polar is given analytically by

$$\mathbf{U}_{polar} = \mathtt{C}\mathbf{P}_{pole}. \tag{23.9}$$

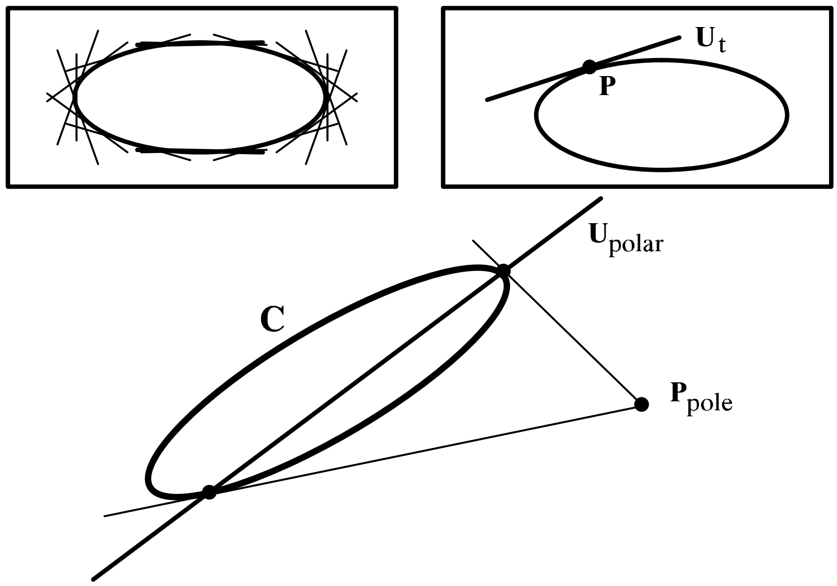

Figure 23.15
The pole and polar relation with respect to a conic. The two contact points at which rays from $\mathbf{P}$ are tangent to the conic, define a line, $\mathbf{U}$, called the polar of $\mathbf{P}$ with respect to C. Conversely, the point $\mathbf{P}$ is the pole of the line $\mathbf{U}$. The polar line of a point on the conic is the tangent at that point, $\mathbf{U}_t$. The dual figure to a conic is the envelope of tangent lines as shown in the upper left figure.

Note that the 3×3 conic matrix defines a correlation between points of the plane and corresponding polar lines. Similarly polar lines map into corresponding poles under the action of $\mathtt{C}^{-1}$. There is not space to show the derivation of equation (23.9), but the case where $\mathbf{P}_{pole}$ is on the conic is easy to demonstrate. In this limiting case the two tangent lines merge into the same line.

The normal to an implicit curve in the projective plane, $f(X_1, X_2, X_3)$, is given by

$$\mathbf{n} = \frac{\partial f}{\partial X_1}\hat{\mathbf{X}}_1 + \frac{\partial f}{\partial X_2}\hat{\mathbf{X}}_2 + \frac{\partial f}{\partial X_3}\hat{\mathbf{X}}_3.$$

We can differentiate, treating each homogeneous coordinate equally, which is equivalent to finding the normal to the surface in ray space defined by $f(X_1, X_2, X_3) = 0$. For a conic curve,

$$A{X_1}^2 + BX_1X_2 + C{X_2}^2 + DX_1X_3 + EX_2X_3 + F{X_3}^2 = 0.$$

The normal vector is

$$\begin{aligned}\mathbf{n} \quad = \quad & \frac{1}{2}[(2AX_1 + BX_2 + DX_3)\hat{\mathbf{X}}_1 + (BX_1 + 2CX_2 + EX_3)\hat{\mathbf{X}}_2 + \\ & (DX_1 + EX_2 + 2FX_3)\hat{\mathbf{X}}_3]\end{aligned}$$

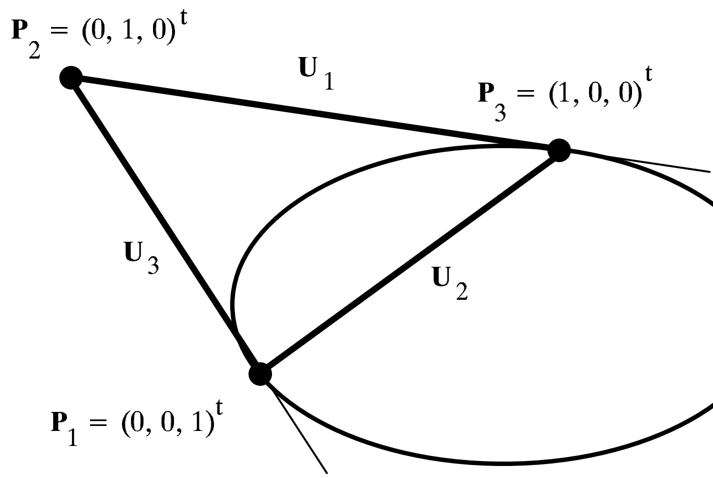

Figure 23.16
A natural triangle of reference for the conic defined by a pole-polar relaton.

where $\hat{\mathbf{X}}_i$ are unit vectors along the homogeneous coordinate axes. The tangent line to the conic at point $\mathbf{P}$ must be perpendicular to the normal and this condition leads to the following line equation,

$$\frac{1}{2}[(2AX_1 + BX_2 + DX_3)X_1 + (BX_1 + 2CX_2 + EX_3)X_2 + (DX_1 + EX_2 + 2FX_3)X_3] = U_1X_1 + U_2X_2 + U_3X_3 = 0. \quad (23.10)$$

The line coefficients $(U_{t1}, U_{t2}, U_{t3})^t$ can be related to the conic matrix by noting this vector is the same as that obtained by multiplying the point vector $(X_1, X_2, X_3)^t$ by the conic matrix, i.e.,

$$\begin{bmatrix} U_{t1} \\ U_{t2} \\ U_{t3} \end{bmatrix} = \begin{bmatrix} AX_1 + \frac{1}{2}BX_2 + \frac{1}{2}DX_3 \\ \frac{1}{2}BX_1 + CX_2 + \frac{1}{2}EX_3 \\ \frac{1}{2}DX_1 + \frac{1}{2}EX_2 + FX_3 \end{bmatrix} = \begin{bmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}.$$

As mentioned above, Equation (23.10) is a correlation if we interpret the conic matrix as a projective transformation matrix. That is the conic coefficient matrix transforms a point, $\mathbf{P}$ on the conic into the tangent line at that point, $\mathbf{U}_t = \mathtt{C}\mathbf{P}$.

In the interpretation of the invariants of conics[19] the concept of a self-polar triangle is important. The definition of self-polar is that each side of the triangle is the polar line of the opposite vertex. One interesting result is that if a self-polar triangle of a conic is used as the triangle of reference then the conic coefficient matrix is diagonal.

[19]See Chapter 3.

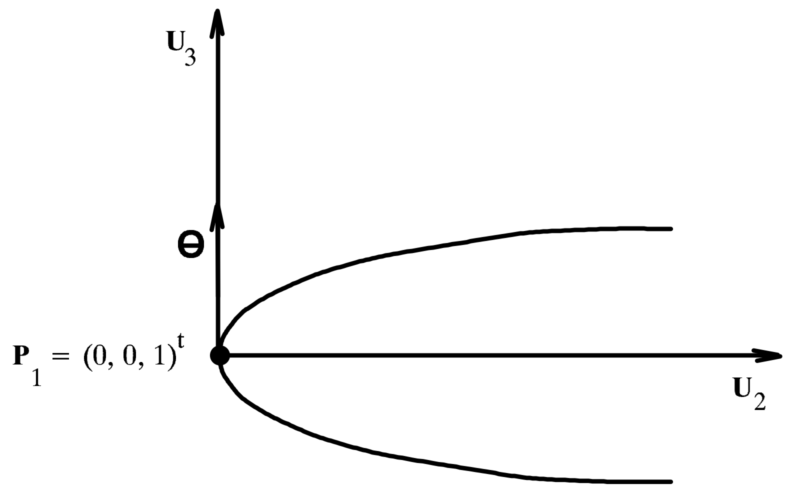

Figure 23.17
The conic in a special triangle of reference defined by the pole-polar relation. The points $\mathbf{P}_2$ and $\mathbf{P}_3$ are ideal points and are shown at infinity. The distance, $x_2 = X_2/X_3 = \theta$, parametrizes the conic.

23.7.2 Parametrizing the conic

Another useful result is illustrated in Figures 23.16 and 23.17. Here the triangle of reference is defined by the pole and polar relation. The pole is assigned the coordinates, $(0, 1, 0)^t$, and the two points of tangency from the pole at the conic are assigned, $(0, 0, 1)^t, (1, 0, 0)^t$. When the conic is represented in this special triangle of reference, it has the form,

$$X_2^2 - X_1X_3 = 0. \tag{23.11}$$

This construction again demonstrates that all conics are projectively equivalent to a parabola, since any triangle can be mapped onto this special triangle of reference by a projective transformation. Also, tangency and incidence are preserved under projection.

An important observation can be seen from this special conic form. Divide equation (23.11) by X_3^2, so $(X_2/X_3)^2 = (X_1/X_3) = \theta^2$. Thus the points on the parabola can be represented parametrically as, $\mathbf{X}(\theta) = (\theta^2, \theta, 1)$. Any other conic can be represented with the same parametrization by applying a projective transformation to the parabola,

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = [\mathrm{T}] \begin{bmatrix} \theta^2 \\ \theta \\ 1 \end{bmatrix}.$$

For example, the unit circle at the origin is defined by,

$$X(\theta) = \begin{bmatrix} (1-\theta^2) \\ 2\theta \\ (1+\theta^2) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta^2 \\ \theta \\ 1 \end{bmatrix}.$$

To see that these projective coordinates describe a unit circle, note that with the substitution, $\theta = \tan\alpha$,

$$x = \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = \cos 2\alpha \qquad y = \frac{2\tan\alpha}{1+\tan^2\alpha} = \sin 2\alpha.$$

Another approach to the parametrization of the conic is to construct a pencil of lines from a fixed point on the conic to any other point on the conic. A single slope parameter, θ, uniquely defines each line of the pencil. This parameter represents the point on the conic which intersects the line of the pencil. As we have seen, the cross-ratio of the lines in a pencil is invariant under projective transformations. Thus the cross-ratio of the parameter values of four points on a conic is also invariant.

By Chasles' theorem, any reference point on the conic defines a fixed cross-ratio with respect to four other points on the conic. Thus the cross-ratio of four points on the conic is invariant to parametrization changes due to the position of the reference point. Since the cross-ratio depends only on the geometric arrangement of the four points on the conic, it is invariant to all parametrizations, i.e., $Cr(\theta_1, \theta_2, \theta_3, \theta_4) = Cr(f(\theta_1), f(\theta_2), f(\theta_3), f(\theta_4))$, where the points of the conic are also specified by $X(\theta) = (f(\theta)^2, f(\theta), 1)^t$.

23.7.3 Circular points

The concept of circular points will prove useful in the relationship between Euclidean and projective transformations. A surprising fact is that all circles intersect the ideal line, $X_3 = 0$, in the same fixed points. A circle is a special case of the general conic with $A = C$ and $B = 0$. Dividing the conic coefficients by A and setting $X_3 = 0$,

$$\left[X_1^2 + X_2^2 + \frac{D}{A}X_1X_3 + \frac{E}{A}X_2X_3 + \frac{F}{A}X_3^2\right]_{X_3=0} = X_1^2 + X_2^2 = 0.$$

This equation has two complex roots, $\mathbf{I} = (1, i, 0)^t$ and $\mathbf{J} = (1, -i, 0)^t$, called the circular points, which are the same for any circle. The key point for vision applications is that any transformation of the projective plane which leaves the circular points fixed is a Euclidean transformation and conversely any Euclidean transformation leaves the circular points fixed. Now a Euclidean transformation is of the form,

$$\mathtt{T}_e = \begin{bmatrix} \cos\theta & \sin\theta & t_x \\ -\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

where θ is a rotation and t_x, t_y represents the translation. Now apply this transformation to the circular point, $\mathbf{I}$.

$$\mathtt{T}\mathbf{I} = \begin{bmatrix} \cos\theta + i\sin\theta \\ -\sin\theta + i\cos\theta \\ 0 \end{bmatrix}.$$

This can be written as,

$$\begin{bmatrix} e^{i\theta} \\ ie^{i\theta} \\ 0 \end{bmatrix} = e^{i\theta} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = \mathbf{I}$$

since all multiples of a projective point are equivalent.

Conversely, it can be shown that any transformation which leaves the circular points fixed is a Euclidean transformation[20]. The easiest way to show this is to write the equations for a line, $\mathbf{U}$, which must pass through both circular points. That is,

$$\begin{aligned} U_1 + iU_2 &= 0 \\ U_1 - iU_2 &= 0. \end{aligned}$$

The equations can be combined to form a composite line conic expression, $U_1^2 + U_2^2 = 0$. This conic represents the double line $x_3 = 0$. We need to show that any transformation which leaves this line conic fixed is a Euclidean transformation. We construct the general projective transformation of the line conic and insist that the result has the same conic coefficients,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathtt{T} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathtt{T}^t.$$

The indicated matrix multiplications impose the following conditions on the elements of $\mathtt{T}$,

$$\begin{aligned} t_{11}^2 + t_{12}^2 &= 1 \\ t_{11}t_{21} + t_{12}t_{22} &= 0 \\ t_{21}^2 + t_{22}^2 &= 1 \\ t_{11}t_{31} + t_{12}t_{32} &= 0 \\ t_{21}t_{31} + t_{22}t_{32} &= 0 \\ t_{31}^2 + t_{32}^2 &= 0 \end{aligned}$$

where we assume that the elements of $\mathtt{T}$ are real. From the last equation it follows that $t_{31} = t_{32} = 0$. The first three equations define the upper

[20]Strictly speaking, the transformation is constrained to be *equiform*, i.e., the entire plane may be scaled uniformly in addition to rotation and translation.

2×2 submatrix of $\mathtt{T}$ to be a rotation. The value of t_{33} is not constrained, so the coordinate axes can be multiplied by an arbitrary scale factor without affecting the conditions. It follows that $\mathtt{T}$ is *equiform*, i.e., a Euclidean transformation with uniform scaling

As we shall see in Section 23.9.1, the concept of circular points generalizes to the concept of the *absolute conic* which is the intersection of all spheres with the ideal plane, $x_4 = 0$, in 3D space. The absolute conic is a key concept for the analysis of camera calibration and camera motion [107]. The invariance of the absolute conic under Euclidean motions of the camera provides a mechanism for using the general machinery of projective geometry while constraining the transformations to correspond to physical camera motions and configurations.

As discussed in Section 23.10.2, camera calibration may be decomposed into two parts, internal camera parameters which describe the geometry of the camera and external parameters which are the six degrees of freedom associated with the position and orientation of the camera reference frame in space. The specification of internal parameters is equivalent to knowing the angle of any ray from the eyepoint through a given point on the image plane, relative to the coordinate frame of the camera. A typical reference direction is the ray, from the eyepoint, perpendicular to the image plane. This ray pierces the image plane at the principal point. The specification of the position of the absolute conic in the image plane is equivalent to knowing the angle between rays, a key component of internal calibration.

The idea can be illustrated by considering the case of a one-dimensional image and perspective projection onto a line as shown in Figure 23.18. The algebra of the one-dimensional case is simple and allows the concepts to be presented in a straightforward manner. Assume that the center of projection is at the origin and the camera axes are aligned with the Cartesian axes of the projective plane. Also assume that $f = 1$. With this camera geometry the circular points project as,

$$
\begin{aligned}
Proj(\mathbf{I}) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} \\
Proj(\mathbf{J}) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}.
\end{aligned}
$$

The Cartesian positions of the circular points are $x_I = 1/i = -i$ and $x_J = 1/(-i) = i$.

If the camera is subjected to a Euclidean transformation in the plane, the position of the circular points in the coordinate frame of the camera is not changed, as we demonstrated earlier. Once the position of the circular points is known, we can determine the angle between rays from

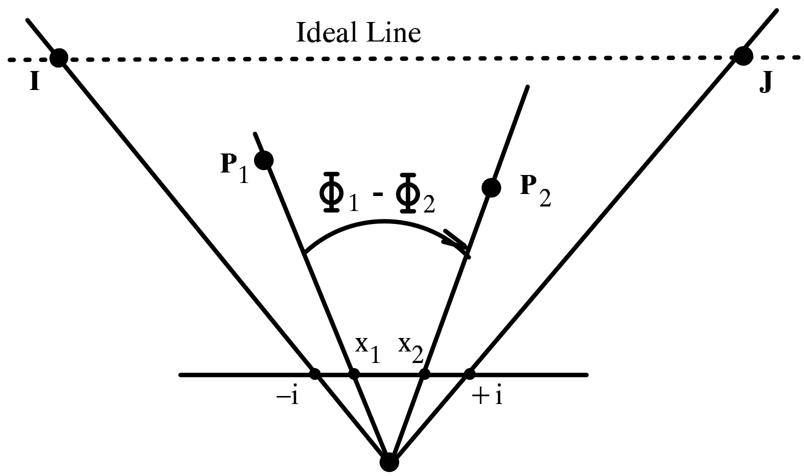

Figure 23.18
The use of circular points to define the angle between lines. The location of the circular points on the ideal line and in the image is independent of Euclidean transformations of the coordinate frame.

the center of projection and passing through any pair of image points. Given the external calibration of a camera, these known angles allow the orientation of a ray to be specified in the plane coordinate frame.

The angle between two rays through image points P_1 and P_2 is given by Laguerre's projective definition of angle [270],

$$e^{i[2(\phi_1-\phi_2)]} = Cr(x_1, x_2, x_I, x_J) = \cos(\phi_1 - \phi_2) + i\sin(\phi_1 - \phi_2)$$

where x_1, x_2 are the Cartesian coordinates of the image points and x_I, x_J are the images of the circular points. For example, take one point at the origin of the image line, $x_1 = 0$ and the other at $x_2 = \sqrt{3}$ which defines a 60^o angle. Then,

$$Cr(0, \sqrt{3}, -i, i) = \frac{-\sqrt{3}+i}{\sqrt{3}+i} = e^{i\frac{2\pi}{3}}$$

or $(\phi_1 - \phi_2) = 60^o$.

The same approach carries over to central projection onto an image plane in three-dimensional Euclidean space. The details of this generalization are given in Section 23.9.1.

23.7.4 The conic and ambiguity of camera calibration

The application of the projective properties of the conic to camera calibration is illustrated by the following simplified example. The standard problem is to compute the parameters of the camera transformation matrix, given a set of image point correspondences with points in 3D space. The parameters for the standard perspective camera are defined in Section 23.10.

We illustrate how the projective properties of the conic simplify the analysis of this problem by considering the case of a one-dimensional camera which images points in the plane onto a line. Projection of points in the plane onto the image line is given by,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}. \tag{23.12}$$

There are five essential parameters of the 2×3 image projection matrix $\mathtt{T}$ since the overall scale is unimportant. Thus, it is necessary to specify five correspondences between points in the plane and points on the image line in order to provide a linear system of equations sufficient to solve for the unknown coefficients of $\mathtt{T}$. A significant issue is to determine if there is any configuration of points in the plane and camera positions where the solution is not unique. Such configurations must be avoided in order to construct a robust camera calibration.

Figure 23.19 shows a set of five reference points. We can select four of the points and the associated image correspondences, say $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4\}$. We know that the cross-ratio of the points on the image line, $Cr(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$, is the same as the cross-ratio of the image rays through the center of projection, $Cr(\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3, \mathbf{U}_4)$. From Chasles' theorem, it follows that the unknown center of projection must lie on a conic, $\mathtt{C}_1$, which passes through the four reference points and defines a constant cross-ratio with respect to the image points. Now consider an independent set of four points, $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_5\}$ which defines a second conic, $\mathtt{C}_2$. The unknown center of projection, $\mathbf{P}_c$, is an intersection point of the two conics. The other three points of intersection are $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\}$. Thus, in most cases, the intersection of the two conics gives an unique solution for $\mathbf{P}_c$. The remaining camera parameters can be determined from the reference points, given the known camera center.

However, by Chasles' theorem, if the center of projection lies on the conic, $\mathtt{C}_r$, uniquely defined by the five reference points, then both $\mathtt{C}_1$ and $\mathtt{C}_2$ are the same conic, $\mathtt{C}_r$. Thus we cannot expect to find a unique solution to the system of equations derived from equation (23.12) when the center of projection and any number of reference points lie on a conic. One can also expect that the solution will be ill-conditioned when the points and center of projection lie close to a conic. This analysis is an example of the use of projective geometry concepts to produce a simple and intuitive picture of problems in photogrammetry and camera motion.

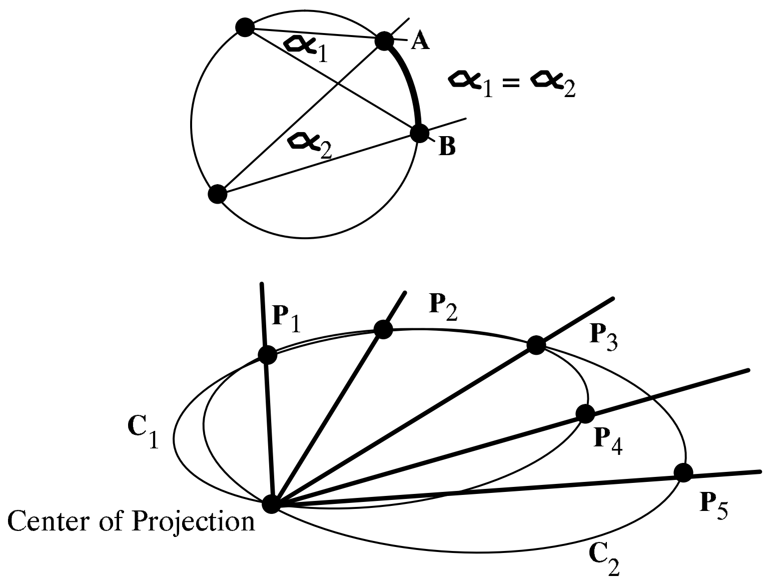

Figure 23.19
A construction to illustrate the condition for ambiguity in camera calibration. When the center of projection of the camera lies on a conic containing all of the reference points, then the camera parameters are ambiguous. The top figure shows that if the angle between rays is known, then the curve of ambiguity is a circle.

A similar development applies for a calibrated camera. The relationship between an image position and the direction of the corresponding ray in space is known for a calibrated camera. In this case only three reference points are needed to determine the unknown position and orientation parameters of the camera coordinate frame. The three reference points define a circle in the plane and when the camera center lies on this circle, the pose parameters cannot be uniquely determined. This ambiguity can be seen from an elementary result from trigonometry; that the angle between lines from a point, $\mathbf{P}$, on a circle to two other points, $\mathbf{A}, \mathbf{B}$ is independent of the position of $\mathbf{P}$, as shown in Figure 23.19. This angle is equal to one half the arc $\overline{\mathbf{AB}}$. Consequently, the three reference points lying on a circle project to the same image positions for any camera with its center of projection anywhere on the circle.

23.8 Non-Euclidean geometry

The concept of the Euclidean distance between two points has no meaning in projective geometry since it is not invariant to transformations of the projective plane. On the other hand, it is reasonable to ask if there is any definition of distance between two points which would hold under

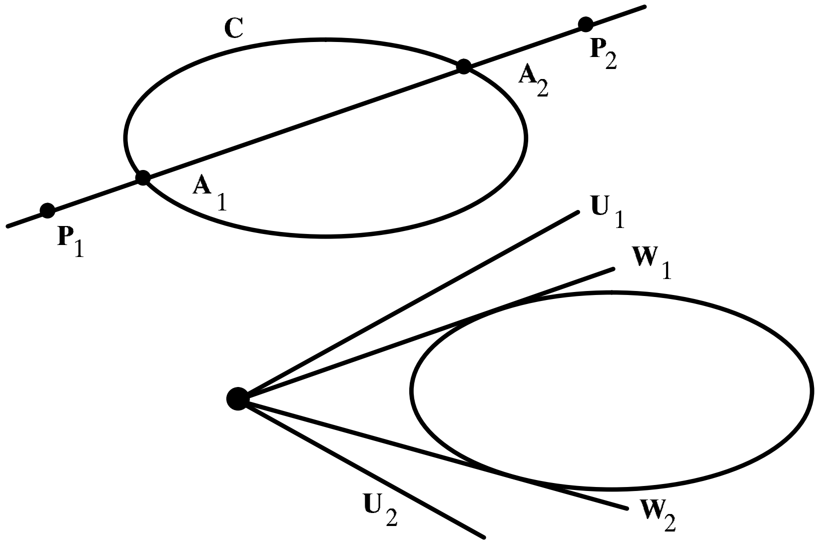

Figure 23.20
A projectively invariant distance can be defined by two points and a conic. In the dual configuration, the conic defines an invariant angle between two lines.

projective transformations.

An invariant distance measure can be defined with respect to a conic. Consider the conic and two points shown in Figure 23.20. Assume for the moment that the line joining the two points, $\mathbf{P}_1, \mathbf{P}_2$, intersects the conic in two real points, $\mathbf{A}_1, \mathbf{A}_2$. Since these four points are collinear the cross-ratio can be used as an invariant distance measure defined as

$$D(\mathbf{P}_1, \mathbf{P}_2) = \log[Cr(\mathbf{P}_1, \mathbf{A}_1, \mathbf{A}_2, \mathbf{P}_2)]. \tag{23.13}$$

$D(\mathbf{P}_1, \mathbf{P}_2)$ is called non-Euclidean distance and has been used in the theory of relativity.

This definition satisfies the following axioms

1. $D(\mathbf{P}_1, \mathbf{P}_3) = D(\mathbf{P}_1, \mathbf{P}_2) + D(\mathbf{P}_2, \mathbf{P}_3)$;
2. $D(\mathbf{P}, \mathbf{P}) = 0$;
3. $D(\mathbf{P}_1, \mathbf{P}_2) + D(\mathbf{P}_2, \mathbf{P}_1) = 0$.

Equation (23.13) can be expressed in terms of the coefficient matrix of the conic, $\mathtt{C}$. The definition depends on the position of the points relative to the conic. When $\mathbf{P}_1, \mathbf{P}_2$ are both inside or both outside the conic the distance is defined by,

$$\cosh^2[D(\mathbf{P}_1, \mathbf{P}_2)] = \frac{(\mathbf{P}_1^t \mathtt{C} \mathbf{P}_2)^2}{(\mathbf{P}_1^t \mathtt{C} \mathbf{P}_1)(\mathbf{P}_2^t \mathtt{C} \mathbf{P}_2)}.$$

If one point is inside the conic and the other outside the conic, then the terms in the denominator are of opposite sign. To keep the result real, it is necessary to introduce a minus sign and use the fact that $\cosh ix = \cos x$,

$$\cos^2[D(\mathbf{P}_1, \mathbf{P}_2)] = -\frac{(\mathbf{P}_1^t \mathtt{C} \mathbf{P}_2)^2}{(\mathbf{P}_1^t \mathtt{C} \mathbf{P}_1)(\mathbf{P}_2^t \mathtt{C} \mathbf{P}_2)}.$$

If all the above properties (given by the axioms) are not needed, then a simple invariant measure of two points is given by

$$I(\mathbf{P}_1, \mathbf{P}_2, \mathtt{C}) = \frac{(\mathbf{P}_1^t \mathtt{C} \mathbf{P}_2)^2}{(\mathbf{P}_1^t \mathtt{C} \mathbf{P}_1)(\mathbf{P}_2^t \mathtt{C} \mathbf{P}_2)}.$$

All of these measures are well defined even if the line joining $\mathbf{P}_1, \mathbf{P}_2$, does not intersect $\mathtt{C}$ in real points.

A similar form gives an invariant measure for the angle between two lines when the conic $\mathtt{C}$ is represented as a line conic, $\mathtt{L} = |\mathtt{C}|\mathtt{C}^{-1}$. Then the angle between two lines, $\mathbf{U}_1, \mathbf{U}_2$, is given by,

$$\theta(\mathbf{U}_1, \mathbf{U}_2) = \log[Cr(\mathbf{U}_1, \mathbf{W}_1, \mathbf{W}_2, \mathbf{U}_2)]$$

where $\mathbf{W}_1, \mathbf{W}_2$ are tangents to the conic from the point of intersection of $\mathbf{U}_1, \mathbf{U}_2$. The cross-ratio of this pencil of four lines gives the invariant angle measure. The angle can be expressed in terms of the line conic coefficients as,

$$\begin{aligned} \cosh^2(\theta) &= \frac{(\mathbf{U}_1^t \mathtt{L} \mathbf{U}_2)^2}{(\mathbf{U}_1^t \mathtt{L} \mathbf{U}_1)(\mathbf{U}_2^t \mathtt{L} \mathbf{U}_2)} \\ \cos^2(\theta) &= -\frac{(\mathbf{U}_1^t \mathtt{L} \mathbf{U}_2)^2}{(\mathbf{U}_1^t \mathtt{L} \mathbf{U}_1)(\mathbf{U}_2^t \mathtt{L} \mathbf{U}_2)} \end{aligned}$$

depending on the sign of the quadratic forms, i.e. the right hand sides of these expressions should always be positive.

These non-Euclidean invariant measures also can be interpreted in terms of the canonical reference frame developed in Section 23.4.7. For example, consider the case of two points, $\mathbf{A}, \mathbf{B}$, and a conic. Each point defines two points of tangency on the conic. The resulting four tangency points can be used to transform the four reference points onto a unit square. The standard Euclidean distance between the transformed points, $\mathbf{a}, \mathbf{b}$, can be used as an invariant distance measure. An advantage of using the non-Euclidean form is that it is not necessary to compute the transform parameters.

Both the non-Euclidean distance and angle forms are directly useful in vision applications. For example, the boundary of the wheel of a car can be used as a reference conic to define invariant measures for any features on the car body which are coplanar with the wheel boundary [118].

23.9 Projective 3D space

23.9.1 Analogies with the projective plane

Many of the results for the projective plane have analogies in three-dimensional space. The idea of homogeneous coordinates can be extended to 3D, where projective points in 3D are represented by $\mathbf{P} = (X_1, X_2, X_3, X_4)$. The following summarizes the major results:

Duality – In projective space, points and planes are dual. The equation of a plane in homogeneous coordinates is:

$$U_1X_1 + U_2X_2 + U_3X_3 + U_4X_4 = 0.$$

Note that the equation is symmetrical in the plane and point coordinates. All *points at infinity* lie on the ideal plane, $X_4 = 0$. Planes through the origin are defined by $U_4 = 0$.

Lines are self-dual entities in 3D projective space and they have a symmetrical relationship with both points and planes. That is, the intersection of two planes define a line, and the join[21] of two points defines a line as well. Note that there is no simple representation of line coordinates in 3D projective space.

Projective transformation – Projective transformations of 3D space are represented as a linear transformation in homogeneous coordinates. Just as before, $\mathbf{x} = \mathtt{T}\mathbf{X}$ where $\mathtt{T}$ is a 4×4 matrix and 15 parameters are needed to specify $\mathtt{T}$ up to a scale factor. A plane is transformed according to, $\mathbf{u} = \mathtt{T}^{-t}\mathbf{U}$ in analogy to the line in the projective plane.

Cross-ratio of planes – The concept of a cross-ratio can be extended to planes in space. In this case, the planes are constrained to have a common line of intersection which again defines a pencil where a single parameter sweeps out the planes. The angle between the planes can be used to define a cross-ratio for four such planes. It can be shown that the cross-ratio of the points of intersection of a line with the pencil of planes is equal to the cross-ratio defined by the angles between the planes [265]. Similarly, if the pencil of planes intersects another plane, the intersection forms a pencil of coplanar and concurrent lines. The cross-ratio of the pencil of these lines of intersection is equal to the cross-ratio of the pencil of planes.

Quadric surfaces – The quadratic form in 3D, $\mathbf{X}^t\mathtt{Q}\mathbf{X} = 0$, corresponds to a quadric surface where $\mathtt{Q}$ is a 4×4 coefficient matrix for the quadric. A quadric is distinguished by being a doubly ruled surface,

[21] The term join means a linear combination of the two points, i.e. any point, $\mathbf{P}$, on the line defined by $\mathbf{P}_1, \mathbf{P}_2$, is given by $\mathbf{P} = \alpha\mathbf{P}_1 + \beta\mathbf{P}_2$.

i.e., in general two distinct lines can be found which lie in the surface and intersect at a given point. The quadric is important in the study of ambiguity in camera motion. It is shown by Maybank [204] that if the center of projection of a camera and any number of reference points lie on a quadric (a hyperboloid of one sheet) then the position of the camera is ambiguous, i.e. there is more than one point on the surface where the camera can be placed so that the image of the reference points is identical. This result is a generalization of the 2D example given in Section 23.7.4.

The absolute conic – There is an analogy to the circular points in the projective plane. All spheres in 3D intersect the ideal plane, $X_4 = 0$, in a conic, called the *absolute conic*. If the 3D space is subjected to a Euclidean transformation, then the absolute conic is fixed. Conversely, any projective transformation of space which leaves the absolute conic fixed is a Euclidean transformation[22]. The equation of a sphere in homogeneous coordinates is a special case of a quadric surface. The intersection of a sphere with the ideal plane, $X_4 = 0$ is,

$$X_1^2 + X_2^2 + X_3^2 = 0.$$

As with the circular points, the conic has only complex points. Many problems in photogrammetry, structure from motion and camera calibration can be analyzed in terms of the absolute conic [107]. For example, the projection of the absolute conic onto an image plane is invariant to Euclidean motions of the camera in space. This fact provides a convenient way of using many of the results of projective geometry while at the same time restricting the analysis to physical camera motion. If the projection of the absolute conic in an image is known, then the angle between rays from the center of projection can be determined from the corresponding image points. This specification of ray angles is a major component of internal camera calibration. The concept is described in more detail in terms of circular points and the case of a central projection onto a line[23].

23.9.2 The twisted cubic

There is a cubic curve in projective 3D space which is an analogous form to the conic in the projective plane. There are a number of interesting properties of the twisted cubic with importance for vision applications. For example, it was pointed out by Buchanan [60] that if the center of projection of a camera and the set of reference points lie on a twisted cubic, then the camera calibration problem does not have a unique solution.

The canonical *twisted cubic* is parametrically defined as

[22] Actually, uniform scaling of the coordinate space is also allowed. See Section 23.7.3.
[23] See Section 23.7.3.

$\mathbf{X}(\theta) = (\theta^3, \theta^2, \theta, 1)^t.$

The equation for a plane incident with a point on the twisted cubic is,

$U_1\theta^3 + U_2\theta^2 + U_3\theta + U_4 = 0$

and has three solutions. It follows that the projection of the twisted cubic is in general a plane cubic curve. That is, a line in the image plane will intersect the projection of the twisted cubic three times, since the line projects to a plane in space and this projected plane intersects the curve in space three times. A simple corollary is that if the center of projection is on the twisted cubic, then the image curve is a conic since one of the three points of intersection is already taken up by the center of projection.

Another important property is that the cross-ratio of the parameters of four points on the twisted cubic is the same as the cross-ratio of the parameters of corresponding points in any projective image of the curve. Thus an invariant descriptor for any set of features which can be associated with a twisted cubic is immediately available. This rather surprising property can be demonstrated with reference to Figure 23.21. First, construct a line, $\mathbf{L}_0$, from the center of projection, $\mathbf{P}_0$ through two points of the twisted cubic[24]. Next, define a pencil of planes with the common line of intersection, $\mathbf{L}_0$. Let the pencil of planes be parametrized by an angle, θ. Since the line $\mathbf{L}_0$ already intersects the cubic in two points, any plane through $\mathbf{L}_0$ must intersect the twisted cubic in a third point, $\mathbf{P}(\theta)$. Thus we can use the pencil of planes to parametrize the twisted cubic in an analogous manner to the construction for the conic in the plane. The cross-ratio of any four points on the twisted cubic is defined by the cross-ratio of the corresponding planes. $\mathbf{L}_0$ changes with viewpoint. However, it can be shown that the cross-ratio has the same value for any $\mathbf{P}_0$ and corresponding $\mathbf{L}_0$[25].

Given this parametrization, we can now demonstrate a remarkable property. Project the twisted cubic from the center of projection, $\mathbf{P}_0$ onto an image plane as shown in Figure 23.21. The line $\mathbf{L}_0$ intersects the image plane at a double point, $\mathbf{l}_0$, of the projected twisted cubic curve, i.e. two points on the twisted cubic map to a single point in the image[26]. The pencil of planes projects to a pencil of lines in the image with $\mathbf{l}_0$ as the common point of intersection. This pencil of lines can be used to parametrize the planar curve since any line of the pencil will intersect the projected curve at only one point. This single point of intersection is the image of the unique intersection point of the plane and the twisted cubic, $\mathbf{P}(\theta)$.

[24] It can be shown that two such points always exist from the general form of the equations defining the points.

[25] See Semple and Kneebone [265] corollary page 301.

[26] There is only one such double point since the existence of two double points would imply that the twisted cubic is a fourth degree curve.

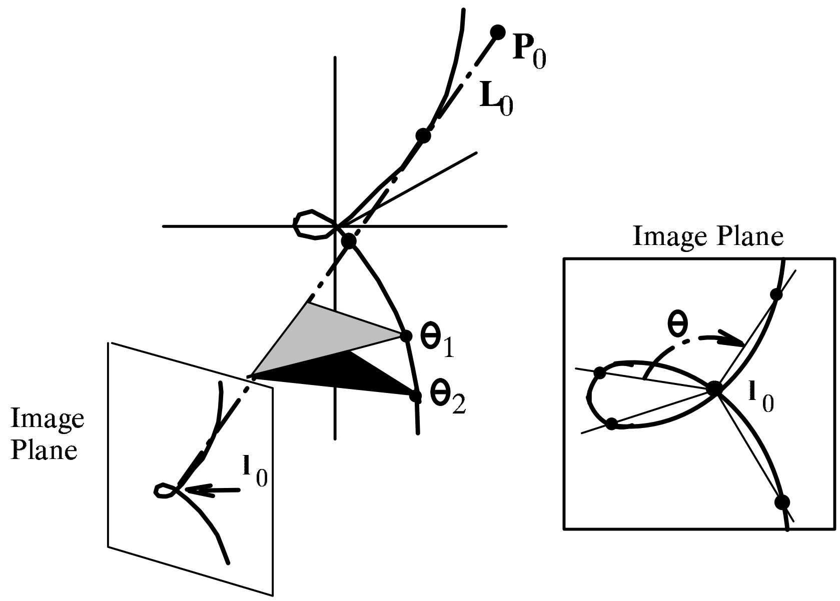

Figure 23.21
A line intersecting the twisted cubic in two points can be used to define a parametrization of the curve in terms of a pencil of planes through the line. A cross-ratio on this pencil is invariant under a projective projection onto an image plane.

Since the cross-ratio of lines in the pencil is the same as the cross-ratio of the pencil of planes in space, we have established the invariance of the cross-ratio on any central projection of the twisted cubic. This result[27] can be used to great advantage in deriving index functions for curved surfaces[28]. The idea is to define generalized cylinders where the axis of the cylinder is a twisted cubic. Then various constructions on the occluding boundary of the object generate distinguished points on the projection of the axis curve. The invariance of the cross-ratio of these distinguished points generates a set of invariant descriptors for the 3D surface.

23.10 The perspective camera

23.10.1 Projective vs perspective image projection

Most of the material up to this point has prepared a framework for the central purpose of the Appendix, the study of the central projection of points in 3D space onto an image plane. The essential geometric

[27]The result holds for any space curve which can be parametrized as $(p(\theta), q(\theta), \theta, 1)$ where p and q are polynomials in θ. Such curves are called curves of genus zero.
[28]See Chapter 11.

properties of this projection can be modeled by the mapping of three-dimensional projective space onto a projective plane, conveniently represented by a linear homogeneous transformation.

A general projective transformation is defined by a 4×4 matrix multiplication,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}.$$

Now, a projection onto a space of one lower dimension can be achieved by simply eliminating one of the coordinates of the transformed projective space. For convenience, we select the plane defined by $x_4 = 0$. That is, all points on the plane can be represented by the homogeneous coordinate vector, $(x_1, x_2, x_3)^t$. The selection of the plane $x_4 = 0$ is general since we can transform any plane to it by a projective transformation of 3D space. The image projection is given by,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \tag{23.14}$$

or $\mathbf{x} = \mathtt{T}\mathbf{X}$. This homogeneous transformation has 11 essential parameters since the overall scale of the matrix does not matter in homogeneous coordinates. It has been demonstrated by Roberts [250] that six or more known reference points in space and the corresponding image points are sufficient to construct a linear system of equations for the 11 unknown parameters of $\mathtt{T}$. Each image point provides two equations for the unknown elements of $\mathtt{T}$ and the solution proceeds in a similar manner to equation (23.4).

The mapping of equation (23.14) can account for many of the geometric aspects of image formation including the case of viewing the projection of a projection, e.g. analyzing the shape of a shadow in an image. The matrix $\mathtt{T}$ can be restricted in form to account for the standard case of projection of 3D space onto an image plane from a single point, i.e. central projection. This restricted model is called a *perspective camera* or the *pinhole camera*. The geometry of the perspective camera is defined in Figure 23.22. In the case of perspective the elements of $\mathtt{T}$ take on a meaning associated with the geometry of central projection. The Euclidean transformation of a point in the world coordinate frame to the camera frame is given by,

$$\mathbf{P}_{cam} = \mathtt{R}(\mathbf{P}_{world} - \mathbf{O})$$

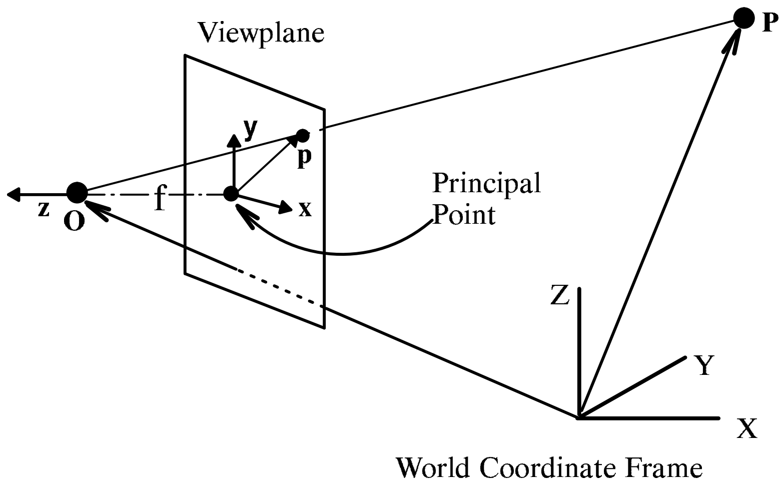

Figure 23.22
The geometry of the perspective camera.

where the matrix,

$$\mathtt{R} = \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

is the rotation matrix from the world coordinate frame to the camera coordinate frame. We make use of the notation, $\mathbf{R}_i$, to indicate each row i of $\mathtt{R}$. $\mathbf{O}$ is the translation vector from the world origin to the camera origin. The origin of the camera is taken to be the center of projection. The transformation is carried out by applying the translation $\mathbf{O}$ followed by the rotation, $\mathtt{R}$.

These two transformations can be applied by a single homogeneous 4×4 transformation matrix,

$$\mathtt{T}_E = \begin{bmatrix} \mathbf{R}_1 & -(\mathbf{R}_1 \cdot \mathbf{O}) \\ \mathbf{R}_2 & -(\mathbf{R}_2 \cdot \mathbf{O}) \\ \mathbf{R}_3 & -(\mathbf{R}_3 \cdot \mathbf{O}) \\ 0\ \ 0\ \ 0 & 1 \end{bmatrix}.$$

Note that each element $\mathbf{R}_i$ represents three elements in a row of the matrix and the last element is a scalar given by the dot product.

Next, the transformed point is projected onto the image plane by the matrix,

$$\mathtt{T}_{Proj} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix}$$

The perpendicular distance from the center of projection to the image plane is called the focal length, f. The point of intersection of the ray from the center of projection, perpendicular to the image plane, is called the principal point. The composite transformation matrix, $\mathtt{T} = \mathtt{T}_{Proj}\mathtt{T}_E$ is,

$$\mathtt{T} = \begin{bmatrix} \mathbf{R}_1 & -(\mathbf{R}_1 \cdot \mathbf{O}) \\ \mathbf{R}_2 & -(\mathbf{R}_2 \cdot \mathbf{O}) \\ \mathbf{R}_3/f & -(\mathbf{R}_3 \cdot \mathbf{O})/f \end{bmatrix}. \tag{23.15}$$

As an illustration, take the case where the center of projection is at the origin and the camera axes are aligned with the world axes. In this case, $\mathtt{T} = \mathtt{T}_{Proj}$, which yields the standard perspective imaging equations used in many vision papers. Given a point in 3D space, $(X, Y, Z, 1)^t$,

$$\begin{aligned} x &= \frac{x_1}{x_3} = f\frac{X}{Z} \\ y &= \frac{x_2}{x_3} = f\frac{Y}{Z}. \end{aligned}$$

It should be noted that the distinction between a perspective transformation and a full projective transformation is that the leftmost 3×3 subarray of $\mathtt{T}$ in equation (23.15) is restricted to be a rotation matrix when the coordinates have been scaled so that $f = 1$.

23.10.2 Intrinsic and extrinsic camera parameters

Equation (23.15) can be rewritten in a form which emphasizes the distinction between parameters which define the internal geometry of the camera and parameters which define the external orientation and position of the camera coordinate frame. That is,

$$\mathtt{T} = \mathtt{T}_{internal}\mathtt{T}_{external} = \begin{bmatrix} s_x & 0 & -t_x \\ 0 & s_y & -t_y \\ 0 & 0 & 1/f \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 & -(\mathbf{R}_1 \cdot \mathbf{O}) \\ \mathbf{R}_2 & -(\mathbf{R}_2 \cdot \mathbf{O}) \\ \mathbf{R}_3 & -(\mathbf{R}_3 \cdot \mathbf{O}) \end{bmatrix}.$$

Here s_x, s_y are image coordinate scale factors and t_x, t_y is the offset of the principal point. Note that the camera focal length parameter is moved into the first 3×3 matrix so that all the internal parameters are represented separately. There are actually only four independent internal parameters since the focal length can be absorbed into the definition of the other elements of $\mathtt{T}_{internal}$. The product of the two matrices maps a point in 3D directly into image pixel coordinates.

Under a general projective image mapping the matrix T has 11 essential parameters. For perspective, the constraints imposed by the camera geometry and the restriction of Euclidean transformation of the camera reference frame reduces the overall degrees of freedom to nine, six for the position and orientation of the camera reference frame and three for the parameters of the image plane in the camera coordinate system. In the discussion above, we have identified 11 distinct camera attributes, i.e., $\{\mathtt{R}, \mathtt{O}, t_x, t_y, s_x, s_y, f\}$. Consequently, these parameters are not all independent in their effect on the perspective transformation and therefore they cannot all be independently recovered from camera calibration.

In a real optical system, additional distortions occur beyond the simple geometric properties of perspective imaging. For example, the image of a point in space is not a point in the image but instead becomes a blur circle due to the finite depth of field of the lens or due to diffraction limits. In addition, a real lens does not produce a uniform mapping across the field of view. These effects are called radial distortion. They become more severe with radial distance from the principal point. It is common to model radial distortion by a low order polynomial as follows,

$$\begin{aligned}
\Delta x &= \bar{x}(K_1 r^2 + K_2 r^4 + \cdots) + [P_1(r^2 + 2\bar{x}^2) + 2P_2\bar{x}\bar{y}][1 + P_3 r^2 + \cdots] \\
\Delta y &= \bar{y}(K_1 r^2 + K_2 r^4 + \cdots) + [P_2(r^2 + 2\bar{y}^2) + 2P_2\bar{x}\bar{y}][1 + P_3 r^2 + \cdots]
\end{aligned}$$

where $\Delta x, \Delta y$ are corrections to the Cartesian image coordinates, $\bar{x} = (x - x_p)$ and $\bar{y} = (y - y_p)$ where $(x_p, y_p)^t$ is the principal point and the radius is defined as $r^2 = \bar{x}^2 + \bar{y}^2$.

The process of measuring the precise geometry of features in the world from image features is called image mensuration. In order to carry out accurate mensuration it is necessary to determine the radial distortion parameters. A good account of a least-mean-squares approach for determining both internal and external camera parameters is given in the Manual of Photogrammetry [267]. It is also the case that accurate calculation of image invariants will require compensation for these radial distortions.

23.10.3 The weak perspective camera

When we consider mappings from 3D space onto a 2D image plane, there is no direct counterpart to the concepts which have been developed for the planar transformations such as Euclidean, affine and projective transformations. However, it is still possible to make useful distinctions among various forms of the 3×4 homogeneous projection matrix, T. We will refer to projections characterized by a 3×4 homogeneous matrix as a *camera*.

Perhaps the most widely used form in vision literature is the *weak perspective* camera. This approximation to perspective viewing has been used in many vision systems [250, 56, 201, 287]. Weak perspective is a limiting form of perspective which occurs when the depth of objects along the line of sight is small compared with the viewing distance. This approximation is carried out as follows. Starting with the general perspective matrix,

$$\mathtt{T} = \begin{bmatrix} \mathbf{R}_1 & -(\mathbf{R}_1 \cdot \mathbf{O}) \\ \mathbf{R}_2 & -(\mathbf{R}_2 \cdot \mathbf{O}) \\ \mathbf{R}_3/f & -(\mathbf{R}_3 \cdot \mathbf{O})/f \end{bmatrix}.$$

Consider a general point in space, $\mathbf{P}$. The transformation of $\mathbf{P}$ is,

$$\mathbf{p} = \mathtt{T}\mathbf{P} = \begin{bmatrix} \mathbf{R}_1 \cdot (\mathbf{P} - \mathbf{O}) \\ \mathbf{R}_2 \cdot (\mathbf{P} - \mathbf{O}) \\ \frac{1}{f}\mathbf{R}_3 \cdot (\mathbf{P} - \mathbf{O}) \end{bmatrix}. \tag{23.16}$$

The distance $d = \mathbf{R}_3 \cdot (\mathbf{P} - \mathbf{O})$ represents the normal distance of the point $\mathbf{P}$ from the image plane, usually called the depth of $\mathbf{P}$. Now given a set of points, $\mathbf{P}_i$, assume that the variation in depth of the points is small compared to the depth of the centroid of the point set, $\mathbf{P}_0$. Let $\mathbf{P}_i = \mathbf{P}_0 + \delta_i$, then

$$\frac{\mathbf{R}_3 \cdot \delta_i}{\mathbf{R}_3 \cdot (\mathbf{P}_0 - \mathbf{O})} << 1.$$

Substituting this approximation into equation (23.16),

$$\mathbf{p}_i = \mathtt{T}(\mathbf{P}_0 + \delta_i) \approx \begin{bmatrix} \mathbf{R}_1 \cdot (\mathbf{P}_i - \mathbf{O}) \\ \mathbf{R}_2 \cdot (\mathbf{P}_i - \mathbf{O}) \\ \frac{1}{f}\mathbf{R}_3 \cdot (\mathbf{P}_0 - \mathbf{O}) \end{bmatrix}.$$

The important result of the approximation is that the homogeneous scale factor is the same for each $\mathbf{p}_i$. That is,

$$\mathbf{p}_i = \frac{f}{\mathbf{R}_3 \cdot (\mathbf{P}_0 - \mathbf{O})}(x_i, y_i, 1)^t = s\ (x_i, y_i, 1)^t$$

where $s = f/[\mathbf{R}_3 \cdot (\mathbf{P}_0 - \mathbf{O})]$. This result is equivalent to multiplying any point in the set by the matrix,

$$\mathtt{T}_{wp} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & -(\mathbf{R}_1 \cdot \mathbf{O}) \\ r_{21} & r_{22} & r_{23} & -(\mathbf{R}_2 \cdot \mathbf{O}) \\ 0 & 0 & 0 & -(\mathbf{R}_3 \cdot (\mathbf{P}_0 - \mathbf{O})/f) \end{bmatrix}.$$

This form can be interpreted as an orthographic projection[29] onto the image plane followed by an isotropic scaling of the image coordinates by s. For this reason, weak perspective is sometimes called scaled orthography. Note that the first two rows of the upper 3×3 matrix of $\mathtt{T}_{wp}$ are the upper rows of a 3D rotation matrix, $\mathtt{R}$. Since $\mathtt{R}$ is orthogonal and of

[29] Orthographic projection is defined in Section 23.10.5.

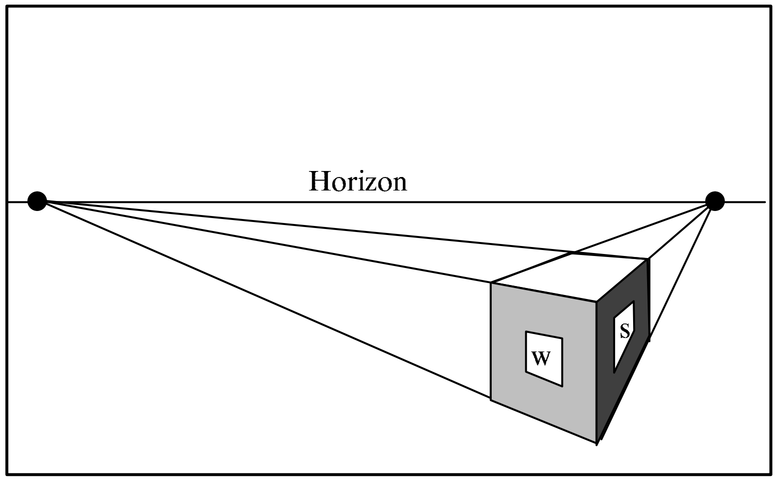

Figure 23.23
An illustration of weak perspective. The face of the cube, **w** is nearly parallel to the observer. Thus, the depth variation along the face is small, compared to the face **s**, and its shape approximates a parallelogram.

unit determinant, it is always possible to recover the missing third row and this provides a basis for deriving object pose from image features. A major property of weak perspective is that parallel lines in space remain parallel in the image.

An illustration of weak perspective is given in Figure 23.23. Note that face of the cube, **w**, which is nearly parallel to the image plane, has less perspective distortion than face **s**, which has a large depth variation along the viewing direction. Another observation is that an image may be *locally* well-approximated by weak perspective but full perspective is required *globally* to accurately represent the entire scene. The total depth variation across the scene can be large, but each individual object may satisfy the weak perspective approximation.

23.10.4 The affine camera

In the development of weak perspective, the approximation was based on a perspective camera and the physical meaning of the weak form of the projection matrix was readily explained, i.e., the depth of an object is small compared to the viewing distance.

The form of the matrix $\mathbf{T}_{wp}$ suggests that a more general class of 3D $\rightarrow$ 2D transformation can be defined where there is no restriction on the form of the elements other than $t_{31} = t_{32} = t_{33} = 0$. We call this more general projection the *affine camera*. Parallelism is still preserved by the affine camera but the image plane shapes are potentially more distorted since the image coordinates can undergo anisotropic scaling. It is not clear what viewing process leads to the affine camera approximation. By analogy to the relationship between perspective and projective transformations in the plane, it may be the case that the affine case results from viewing a weak perspective image with a weak perspective camera. In any case, the affine camera form is useful in developing invariants of 3D transformation groups which can be recovered from multiple views with uncalibrated cameras, as in Section 1.6.3. Allowing for unknown internal camera parameters is equivalent to the extra degrees of freedom of the affine camera.

23.10.5 Orthographic projection

Orthographic projection results from the limit where the rays from the center of projection are parallel. This limit can be represented by letting the focal length approach infinity while keeping the scale factor at unity. The form of the perspective transformation matrix becomes,

$$\mathtt{T}_{orth} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & -(\mathbf{R}_1 \cdot \mathbf{O}) \\ r_{21} & r_{22} & r_{23} & -(\mathbf{R}_2 \cdot \mathbf{O}) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The main difference between orthographic projection and weak perspective is that distances along directions parallel to the image plane are preserved under orthography. For this reason, orthographic projections are used extensively in mechanical drawing to define the 3D dimensions of objects.

23.10.6 Mapping between planes

The general projective or perspective transformation matrix can be used to specify the mapping between two planes in space. Here the points in space $\mathbf{X}_i$ are assumed to lie on a plane. Without loss of generality, it can be assumed that the first plane corresponds to the X, Y plane of the world coordinate system and the second plane is the image plane. That is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix}.$$

So,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{14} \\ t_{21} & t_{22} & t_{24} \\ t_{31} & t_{32} & t_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}.$$

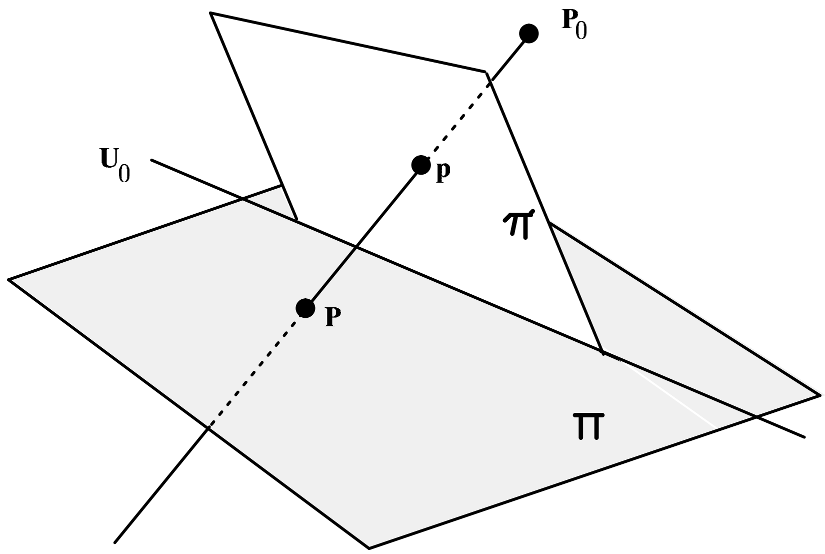

Figure 23.24
The perspective mapping between two planes. Note that line $\mathbf{U}_0$ is fixed under the perspectivity.

So we see that the general projective mapping between planes in space is specified by a 3×3 homogeneous transformation as introduced in Section 23.4.5. The elements of the 3×3 matrix correspond to the first, second, and fourth columns of the original matrix $\mathtt{T}$.

The perspective mapping between two planes is a central projection from a single point in space where corresponding points in the planes are collinear with the center of projection. In the case of perspective mapping it is observed that the first two columns of the 3×3 matrix must be orthogonal and have the same norm in a coordinate frame where $f = 1$. With these restrictions, a theory of the perspective plane can be developed in analogy to the results of Section 23.4. These perspective transformations between planes are called *perspectivities*. The geometry of a perspectivity is shown in Figure 23.24. The line of intersection between the two planes is fixed under the perspectivity. It is emphasized that perspective mappings of the plane do not form a group since the composition of two perspectivities is not in general a perspective transformation, i.e./ the special form of the perspective matrix is not in general preserved by the product of two such matrices.

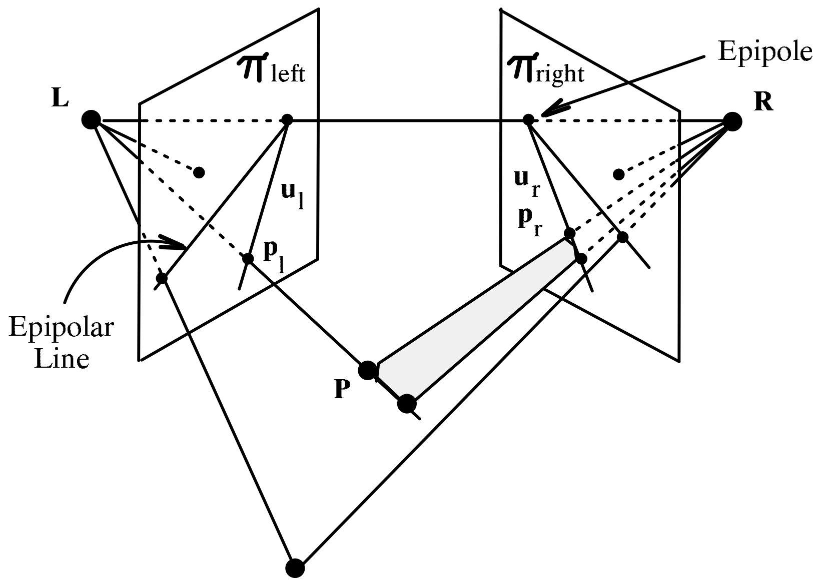

Figure 23.25
The geometry of two perspective views. Lines $\mathbf{u}_l, \mathbf{u}_r$ are called epipolar lines.

23.11 Multiple views

It is emerging[30], that a rich theory of invariants can be developed in connection with multiple views of 3D space. The geometry of two arbitrary perspective views is shown in Figure 23.25. The line joining the centers of projection of each view, $\mathbf{T} = \overline{\mathbf{LR}}$, intersects each image plane in a point called the epipole. Given a point in space, $\mathbf{P}$, the plane, $[\mathbf{L}, \mathbf{R}, \mathbf{P}]$, intersects each image in a line called the epipolar line. The significance of the epipolar line is seen by fixing the image of $\mathbf{P}$ in the left image, $\mathbf{p}_l$. The center of projection, $\mathbf{L}$, and $\mathbf{p}_l$ defines a ray, $\overline{\mathbf{Lp}_l}$. That is, the point in space, $\mathbf{P}$, can lie anywhere on this ray. The image of the ray in the right image plane is just the epipolar line, $\mathbf{u}_r$.

Corresponding 3D points, $\{\mathbf{P}_l, \mathbf{P}_r\}$ in the left and right views can be related as follows [198, 238]. The left and right coordinate frames are related by a Euclidean transformation,

$$\mathbf{P}_r = \mathtt{R}\mathbf{P}_l + \mathbf{T}$$

Now in general, $\mathbf{P}_l \cdot (\mathbf{T} \times \mathbf{P}_r) = 0$, for any vectors $\mathbf{P}, \mathbf{T}$. The cross product by $\mathbf{T}$ is equivalent to a matrix multiplication by,

$$\tau = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}.$$

[30]See Chapters 14 and 15.

Thus the vector identity can be written as,

$$\mathbf{P}_r^t \; \mathtt{E} \; \mathbf{P}_l = 0.$$

Where $\mathtt{E} = \tau \mathtt{R}$ is called the essential matrix. The image points in homogeneous coordinates are proportional to the 3D Cartesian point coordinates. Without loss of generality, assume that the right image coordinate frame is aligned with the world coordinate frame, then,

$$\begin{bmatrix} x_r \\ y_r \\ 1 \end{bmatrix} = \frac{1}{Z_r} \begin{bmatrix} X_r \\ Y_r \\ Z_r \end{bmatrix}$$

and similarly for the point in the left image. Thus, we have the following constraint on the homogeneous coordinates on corresponding points in the left and right images

$$\mathbf{p}_r^t \; \mathtt{E} \; \mathbf{p}_l = 0. \tag{23.17}$$

The same form of equation holds even if the two image views are full projective transformations. A full projective camera can be formed by applying a planar projective transformation to a perspective camera image. Assume that each image is transformed by planar projective transformations, $\mathbf{p}_l = \mathtt{T}_l \mathbf{p}_l'$, $\mathbf{p}_r = \mathtt{T}_r \mathbf{p}_r'$, then equation (23.17) becomes,

$$\mathbf{p}_r'^{\,t} \; \mathtt{T}_r \; \mathtt{E} \; \mathtt{T}_l \; \mathbf{p}_l' = \mathbf{p}_r'^{\,t} \; \mathtt{Q} \; \mathbf{p}_l' = 0 \tag{23.18}$$

so that the original result of Longuet-Higgins, equation (23.17), can be extended to two arbitrary projective (rather than perspective) image projections of a 3D point set.

The epipolar relation is derived from equation (23.18) by rewriting it as,

$$\mathbf{u}_r^t \; \mathbf{p}_r = 0.$$

Thus equation (23.18) is just the equation for a line with the line coordinates given by $\mathbf{u}_r = \mathtt{Q} \; \mathbf{p}_l$. So the matrix $\mathtt{Q}$ maps points in the left image onto lines in the right image. This mapping has a similar form to a correlation but it is not a true correlation because $\mathtt{Q}$ is singular and has no inverse. It is seen that $\mathtt{Q}$ must be singular since all the epipolar lines meet at the epipole and are thus linearly dependent.

The matrix $\mathtt{Q}$ just defined can be used to develop a transfer of features seen in two images to a third view. The basic idea of *transfer* using invariants was developed in Section 23.5 in the context of the cross-ratio. For the case of two views, with eight point correspondences, it is possible to transform any point seen in each of the views to a third image. No camera information is assumed for any of the views. The approach is illustrated in Figure 23.26. It is assumed that eight points

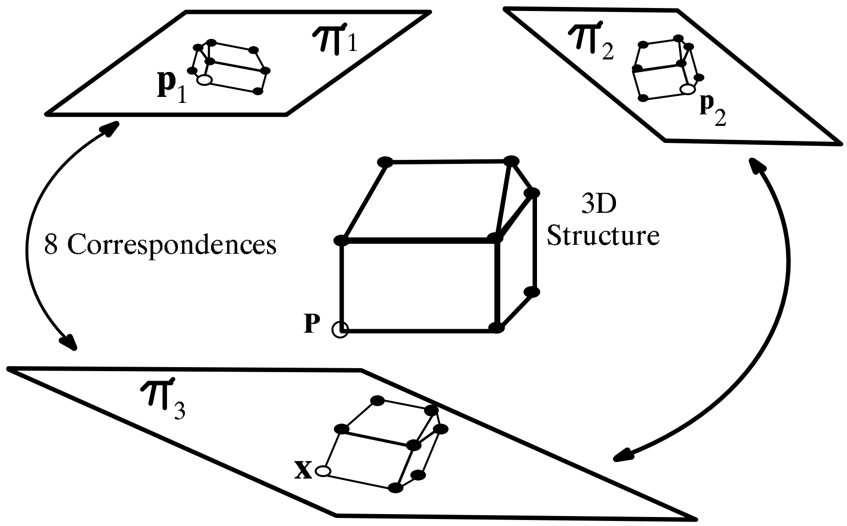

Figure 23.26
Using the invariant of two views to transfer a model to a third view. Given eight correspondences between view 1 and view 3 as well as between view 2 and view 3, any other geometric features can be transferred to the correct shape and position in view 3.

can be identified between π_a and π_c as well as between π_b and π_c. As shown by Barrett in Chapter 14, the image position of any ninth point, $\mathbf{x}$ can be found using an invariant of two views which we now derive from the form of $\mathtt{Q}$. We will show that an invariant can be defined for two views, on eight reference points, which leads to a transfer procedure analogous to that described in Section 23.5.5. Expanding the general form, $\mathbf{p}^t\mathtt{Q}\mathbf{p} = 0$,

$$\mathbf{p}_l^t\mathtt{Q}\mathbf{p}_r = \begin{bmatrix} x_l & y_l & 1 \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ 1 \end{bmatrix}.$$

This result further expands to,

$$\begin{aligned} & x_l x_r q_{11} + x_l y_r q_{12} + x_l q_{13} + \\ & y_l x_r q_{21} + y_l y_r q_{22} + y_l q_{23} + \\ & \qquad x_r q_{31} + y_r q_{32} + q_{13}. \end{aligned}$$

This expansion can also be written as a dot product, $\mathbf{b} \cdot \mathbf{q}$, where

$$\begin{aligned} \mathbf{b} &= (x_l x_r, x_l y_r, x_l, y_l x_r, y_l y_r, y_l, x_r, y_r, 1)^t \qquad \text{and} \\ \mathbf{q} &= (q_{11}, q_{12}, q_{13}, q_{21}, q_{22}, q_{31}, q_{32}, q_{33})^t. \end{aligned}$$

Given nine points correspondences between the left and right images, a 9×9 matrix, $\mathtt{B}$, can be constructed as follows,

$$\mathtt{B} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \dots \\ \mathbf{b}_9 \end{bmatrix}.$$

From the fact that $\mathbf{p}^t \mathtt{Q} \mathbf{p} = 0$, it follows that $\mathtt{B}\mathbf{q} = 0$. In order for this linear system of equations to have a non-trivial solution for $\mathbf{q}$, the determinant of $\mathtt{B}$ must be identically zero. The condition $|\mathtt{B}| = 0$ is an invariant for two views since it holds for any position of the cameras and any selection of the point-set.

This invariant can be used for transfer by selecting eight points and then transferring any ninth point. Let the transferred point be denoted by $\mathbf{x}$. The vector $\mathbf{b}$ is a quadratic form in two points which we make more explicit by the notation $\mathbf{b}(\mathbf{p}_i, \mathbf{p}_j)$. Now referring to Figure 23.26 we can write the form for $\mathtt{B}$ between views 1 and 2 as,

$$\mathtt{B} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \dots \\ \mathbf{b}(\mathbf{p}_1, \mathbf{x}) \end{bmatrix}$$

where $\mathbf{x}$ denotes the position of $\mathbf{x}$ in view 3. Note that the invariant condition $|\mathtt{B}| = 0$ leads to a linear expression in terms of the coordinates of $\mathbf{x}$. That is, $|\mathtt{B}| = 0$, can be expanded as

$$\alpha x_x + \beta y_x + \gamma = 0.$$

Thus, given eight point correspondences between views 1 and 3, the position of $\mathbf{x}$ lies on a line in image 3. Similarly, another line is defined by the invariant condition between views 2 and 3. The intersection of these two lines gives the position of $\mathbf{x}$. Any features such as lines and curves can be transferred in this manner once eight point correspondences are available.

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