Instructor: Xiaodi Wu
Date: prepared by Yiming Huang, Xiaodi Wu

Exercises on Linear Algebra (Optional)

Here is a list of example problems on the linear algebra that we will use in quantum information. You **don't** need to submit solutions for these problems, However, you probably want to figure out answers to these problems as well.

Problem 1.1. For complex number $c = a + bi$, recall that the real and imaginary parts of c are denoted $\mathrm{Re}(c) = a$ and $\mathrm{Imag}(c) = b$.

- Prove that $c + c^* = 2 \cdot \mathrm{Re}(c)$.
- Prove that $cc^* = a^2 + b^2$.
- What is the polar form of $c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$? Use the fact that $e^{i\theta} = \cos\theta + i\sin\theta$?
- Draw $c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ as a vector in the complex plane.

Solution 1.1.

- Prove that $c + c^* = 2 \cdot \mathrm{Re}(c)$.
  Proof:

$$
\begin{aligned}
c + c^* &= (a + bi) + (a - bi) \\
&= 2 \cdot \mathrm{Re}\,(c)
\end{aligned}
$$

- Prove that $cc^* = a^2 + b^2$.
  Proof:

$$
\begin{aligned}
cc^* &= (a + bi) \times (a - bi) \\
&= a^2 + abi - abi - i^2b^2 = a^2 + b^2
\end{aligned}
$$

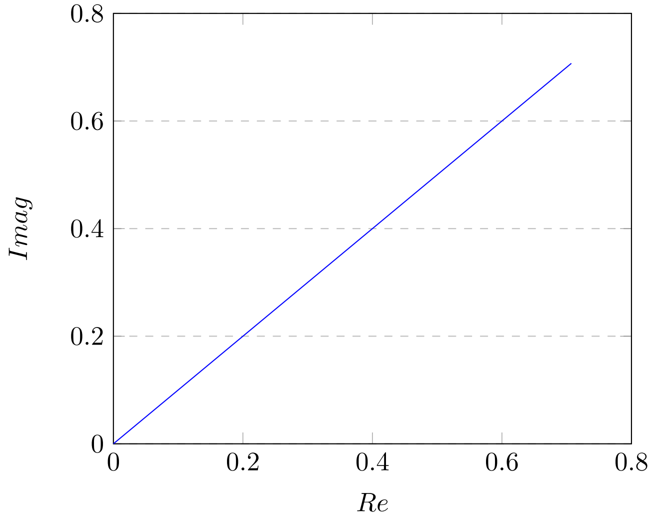
- What is the polar form of $c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$? Use the fact that $e^{i\theta} = \cos\theta + i\sin\theta$?
  Proof:

$$
\because e^{i\theta} = \cos\theta + i\sin\theta = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i
$$

$$
\cos\theta = \frac{1}{\sqrt{2}}, \sin\theta = \frac{1}{\sqrt{2}}, \theta = \frac{\pi}{4} + 2k\pi, k \in \mathbb{N}
$$

$$
\therefore e^{i\theta} = e^{i\left(\frac{\pi}{4}\right)}
$$

- Draw $c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ as a vector in the complex plane.



Problem 1.2. Define that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

- What is $\text{tr}(A|1\rangle\langle 0|)$? (Hint: This can be computed quickly by using the cyclic property of the trace and the outer product representation of A. Do master this trick; it will be used repeatedly in the course and save you much time.)
- Let $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Use the same trick above, along with the fact that the trace is linear, to quickly evaluate

$$\text{tr}(A \cdot |+\rangle\langle +|).$$

Solution 1.2.

- b
- $\frac{1}{4}(a+b+c+d)$

Problem 1.3.

- Write out the 4-dimensional vector for $(\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$?
- Let $\mathcal{B}_1 = \{|\psi_1\rangle, |\psi_2\rangle\}$, $\mathcal{B}_2 = \{|\phi_1\rangle, |\phi_2\rangle\}$ be two orthonormal bases for $\mathbb{C}^2$. Can you construct an orthonormal basis for $\mathbb{C}^4$?

Solution 1.3.

- $\begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}$

- $B_3 = \{|\psi_1\rangle \otimes |\phi_1\rangle, |\psi_1\rangle \otimes |\phi_2\rangle, |\psi_2\rangle \otimes |\phi_1\rangle, |\psi_2\rangle \otimes |\phi_2\rangle\}$

Problem 1.4.

- Write out the 4×4 matrix representing $Y \otimes Y$.
- Prove that $(Z \otimes Y)^\dagger = Z \otimes Y$. Do not write out any matrices explicitly; rather, you must use the properties of the tensor product, dagger, and Y.

Solution 1.4.

- $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$
- $\because (Z \otimes Y)^\dagger = Z^\dagger \otimes Y^\dagger$ and$Z^\dagger = Z, Y^\dagger = Y \therefore (Z \otimes Y)^\dagger = Z \otimes Y$

Problem 2.1.

- Write down a matrix that is not Hermitian.
- Let $A \in \mathrm{L}\left(\mathbb{C}^d\right)$ be a Hermitian matrix. Prove that if for all $|\psi\rangle \in \mathbb{C}^d$, $\langle\psi| A |\psi\rangle \geq 0$, then A has only non-negative eigenvalues.
- Let $A \in \mathrm{L}\left(\mathbb{C}^d\right)$ be a Hermitian matrix. Prove that if A has only non-negative eigenvalues, then for all $|\psi\rangle \in \mathbb{C}^d$, $\langle\psi| A |\psi\rangle \geq 0$.

Solution 2.1.

- $$\begin{pmatrix} 1 & i \\ 1-i & 1 \end{pmatrix}$$
- Let $|\psi\rangle$ be the eigenvector of A
$$\begin{aligned} &\because A = A^\dagger, \langle\psi| A |\psi\rangle \geqslant 0 \\ &\therefore \langle\psi| A |\psi\rangle = \langle\psi| (A |\psi\rangle) = \lambda \langle\psi \mid \psi\rangle = \lambda \geqslant 0 \end{aligned} \tag{1}$$
- Given A is Hermitian, let $|\psi_i\rangle$ be A's eigenvector with eigenvalue $\lambda_i \geq 0$ for $i \in [d]$. Thus, for any $|\psi\rangle \in \mathbb{C}^d$, we have
$$|\psi\rangle = \sum_i \alpha_i |\psi_i\rangle .$$
Then we have
$$A|\psi\rangle = \sum_i \alpha_i A |\psi_i\rangle = \sum_i \alpha_i \lambda_i |\psi_i\rangle .$$
Thus,
$$\langle\psi| A |\psi\rangle = \sum_i |\alpha_i|^2 \lambda_i \geq 0.$$

Problem 2.2. Given $|-\rangle$ state, and suppose that we measure in the computational basis $B = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$. What are the probabilities for each possible measurement outcome, and the corresponding post-measurement states?

Solution 2.2. The probability of outcome 0 with post-measurement state $|0\rangle$ is 0.5, and the probability of outcome 1 with post-measurement state $|1\rangle$ is also 0.5.

Problem 2.3.

- Let $A, B \in \mathrm{L}\left(\mathbb{C}^d\right)$ be positive semi-definite matrices. Prove that $A + B$ is positive semi-definite.
- Prove that if ρ and σ are density matrices, then so is $p_1\rho + p_2\sigma$ for any $p_1, p_2 \geq 0$ and $p_1 + p_2 = 1$.

Solution 2.3.

- For any $|\psi\rangle$, we have

$$\begin{aligned} &\because \langle\psi| A |\psi\rangle \geqslant 0, \langle\psi| B |\psi\rangle \geqslant 0 \\ &\therefore \langle\psi| (A+B) |\psi\rangle = \langle\psi| A |\psi\rangle + \langle\psi| B |\psi\rangle \geqslant 0 \end{aligned} \tag{2}$$

- It suffices to show that $p_1\rho + p_2\sigma$ is positive semi-definite (which is basically implied by the first item) and has trace 1. The later follows from

$$\mathrm{tr}(p_1\rho + p_2\sigma) = p_1 \mathrm{tr}(\rho) + p_2 \mathrm{tr}(\sigma) = p_1 + p_2 = 1.$$

Problem 2.4. Let $|\psi\rangle = \alpha_0 |a_0\rangle |b_0\rangle + \alpha_1 |a_1\rangle |b_1\rangle$ be the Schmidt decomposition of a two-qubit state $|\psi\rangle$. Prove that for any single qubit unitaries U and V, $|\psi\rangle$ is entangled if and only if $|\psi'\rangle = (U \otimes V) |\psi\rangle$ is entangled.

Solution 2.4. This is almost by definition. Try to formalize the argument.
