# CMSC 250
# Discrete Structures

Spring 2019

# Recall Conditional Statement

**Definition**

A sentence of the form "If $p$ then $q$" is symbolically donoted by p→q

**Example**

If you show up for work Monday morning, then you will get the job.

| **p** | **q** | **p → q** |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

# Definitions for Conditional Statement

**Definition**

The *contrapositive* of $p \to q$ is $\sim q \to \sim p$

The *converse* of $p \to q$ is $q \to p$

The *inverse* of $p \to q$ is $\sim p \to \sim q$

# Contrapositive

**Definition**

The *contrapositive* of a conditional statement is obtained by transposing its conclusion with its premise and inverting. So, Contrapositive of $p \to q$ is $\sim q \to \sim p$.

**Example**

Original statement: *If I live in Denver, then I live in Colorado*.

Contrapositive: *If I don't live in Colorado, then I don't live in Denver*.

**Theorem**

*The contrapositive of an implication is equivalent to the original statement*

# Converse

## Definition

The *Converse* of a conditional statement is obtained by transposing its conclusion with its premise. So, Converse of $p \to q$ is $q \to p$.

## Example

Original statement: *If I live in Denver, then I live in Colorado*.

Contrapositive: *If I live in Colorado, then I live in Denver*.

The converse is **NOT** logically equivalent to the original.

# Inverse

**Definition**

The *Inverse* of a conditional statement is obtained by inverting its premise and conclusion. So, inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$.

**Example**

Original statement: *If I live in Denver, then I live in Colorado*.
Contrapositive: *If I do not live in Denver, then I do not live in Colorado*.

The inverse is **NOT** logically equivalent to the original.

# Biconditional Statements (If and only if)

**Example**

- I will carry my umbrella, if and only if it is raining.
- I am breathing, if and only if I am alive.

| p | q | p↔q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

# Necessary and sufficient conditions

If $p$ and $q$ are statements

**Definition**

p is a sufficient condition for q means, if p then q

p is a necessary condition for q means, if not p then not q

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p is a necessary condition for q also means if q then p.

# Necessary and sufficient conditions

If $p$ and $q$ are statements

**Definition**

p is a sufficient condition for q means, if p then q

p is a necessary condition for q means, if not p then not q

p is a necessary condition for q also means if q then p.

As a consequence,
p is a necessary and sufficient condition for q means,
p if, and only if, q.

# Necessary and sufficient conditions

**Example**

If Jane is eligible to vote, then she is at least 18 years old.

# Arguments

**Definition**

An *argument* is a conjecture that says:
If you make certain assumptions, then a particular statement must follow.

- The assumptions are called *premises* or *hypotheses*
- The statement that follows is the *conclusion*

# Arguments

**Example**

If you have a current password, then you can log onto the network
You have a current password.
Therefore,
You can log onto the network

$$\begin{array}{l} p \rightarrow q \\ \underline{p \qquad} \\ \therefore q \end{array}$$

# Validity

**Definition**

An *argument* is valid when,
for all interpretations that make the premises true, the conclusion is also true.

Is this argument valid?

P1: $p \vee q$

P2: $q \rightarrow r$

P3: $\sim p$

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$\therefore r$

# Laws of logical equivalencies

| Given any statement variables $p$, $q$, and $r$, a tautology $t$ and a contradiction $c$, the following logical equivalences hold: | | |
|---|---|---|
| 1. Commutative laws: | $p \wedge q \equiv q \wedge p$ | $p \vee q \equiv q \vee p$ |
| 2. Associative laws: | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | $(p \vee q) \vee r \equiv p \vee (q \vee r)$ |
| 3. Distributive laws: | $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ |
| 4. Identity laws: | $p \wedge t \equiv p$ | $p \vee c \equiv p$ |
| 5. Negation laws: | $p \vee \neg p \equiv t$ | $p \wedge \neg p \equiv c$ |
| 6. Double Negative law: | $\neg(\neg p) \equiv p$ | |
| 7. Idempotent laws: | $p \wedge p \equiv p$ | $p \vee p \equiv p$ |
| 8. DeMorgan's laws: | $\neg(p \wedge q) \equiv \neg p \vee \neg q$ | $\neg(p \vee q) \equiv \neg p \wedge \neg q$ |
| 9. Universal bounds laws: | $p \vee t \equiv t$ | $p \wedge c \equiv c$ |
| 10. Absorption laws: | $p \vee (p \wedge q) \equiv p$ | $p \wedge (p \vee q) \equiv p$ |
| 11. Negations of t and c: | $\neg t \equiv c$ | $\neg c \equiv t$ |

# Rules of inference

## Definition

*Rules of Inference* are short arguments that are known to be valid. We will use them to prove the validity of more complex arguments.

| Modus Ponens | Modus Tollens | Conjunction | Transitivity |
|---|---|---|---|
| $p \to q$<br>$p$<br>$\therefore q$ | $p \to q$<br>$\sim q$<br>$\therefore \sim p$ | $p$<br>$q$<br>$\therefore p \wedge q$ | $p \to q$<br>$q \to r$<br>$\therefore p \to r$ |

| Elimination | | Generalization | |
|---|---|---|---|
| $p \vee q$<br>$\sim q$<br>$\therefore p$ | $p \vee q$<br>$\sim p$<br>$\therefore q$ | $p$<br>$\therefore p \vee q$ | $q$<br>$\therefore p \vee q$ |

| Specialization | | Contradiction rule | Proof by division into cases |
|---|---|---|---|
| $p \wedge q$<br>$\therefore p$ | $p \wedge q$<br>$\therefore q$ | $\sim p \to c$<br>$\therefore p$ | $p \vee q$<br>$p \to r$<br>$q \to r$<br>$\therefore r$ |

# Modus Ponens and Modus Tollens Examples

## Modus Ponens

| | |
|---|---|
| If it is sunny, it is hot | $p \rightarrow q$ |
| It is sunny | $p$ |
| $\therefore$ it is hot. | $\therefore q$ |

## Modus Tollens

| | |
|---|---|
| If it is sunny, it is hot | $p \rightarrow q$ |
| It is not hot | $\sim q$ |
| $\therefore$ it is not sunny. | $\therefore \sim p$ |

# Practice

**Example**

If there are more pigeons than there are pigeonholes, then at least two pigeons roost in the same hole.

There are more pigeons than there are pigeonholes

$\therefore$

# Practice

**Example**

If there are more pigeons than there are pigeonholes, then at least two pigeons roost in the same hole.
There are more pigeons than there are pigeonholes
$\therefore$ at least two pigeons roost in the same hole.

# Practice

**Example**

If 870,232 is divisible by 6, then it is divisible by 3.

870,232 is not divisible by 3

$\therefore$

# Practice

**Example**

If 870,232 is divisible by 6, then it is divisible by 3.

870,232 is not divisible by 3

$\therefore$ 870232 is not divisible by 6.

# Practicing Formal Proof

**Example**

P1: $p \vee q$

P2: $q \rightarrow r$

P3: $\sim p$

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$\therefore$ r

# Practicing Formal Proof

Example

*Hypotheses*

It is not sunny this afternoon and it is colder than yesterday.

We will go swimming only if it is sunny.

If we do not go swimming, then we will take a canoe trip.

If we take a canoe trip, then we will be home by sunset.

*Conclusion*

We will be home by sunset.