CMSC 250
Discrete Structures

Predicate Logic

Propositional logic has limitations

Definition

All men are mortal
Socrates is a man

$\therefore$ Socrates is mortal

Predicates

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A *Predicate* is a sentence (containing variables) that is either true or false depending on the values substituted for the variables.

Example

Karim is a student at University of Maryland.

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Karim is a student at University of Maryland.

The word “Karim” is a subject and “is a student at University of Maryland” is a predicate

Predicates in logic

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To obtain predicate, we can remove one or more nouns

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P: is a student at University of Maryland

Q: is a student at

Both P and Q are predicate symbols.

Predicate Symbols

Example

Predicate	Domain (set)
Q(x): x is even	x ϵ $\mathbb{Z}$
P(x, y, z): $x^2 + y^2 = z^2$	x, y, z ϵ $\mathbb{R}$
R(a,b): a is a factor of b	a, b ϵ $\mathbb{N}$
S(a, b): a is taller than b	a,b ϵ students in this class
G(Y): y is green	y ϵ M&M
A(c): c is under attack	c ϵ Computers

Logical Connectives

Definition

The logical connectives can be used to join predicates to make more complex predicates.

Examples

- $P(x) = \sim Q(x) \vee R(x)$
- $T(x,y) = (A(x) \wedge G(x,y)) \rightarrow \sim L(y)$

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- We bind the variables using quantifiers, to find out,
  - Whether the claim applies to all values of the variable - universal quantification
  - whether it may only apply to some - existential quantification.

Universal Quantifier

Definition

The *Universal quantifier*, $\forall$ (for all), says that a statement must be true for all values of a variable.

Example

- All humans are mortal
  $$\forall x\text{: Human}(x) \rightarrow \text{Mortal}(x)$$
- If x is positive then $x + 1$ is positive
  $$\forall x : x > 0 \rightarrow x + 1 > 0$$

More Universal Quantifier

Definition

To make the universe of the values explicit, we use set membership notation

Example

$\forall x \ \epsilon \ \mathbb{Z} : x > 0 \rightarrow x + 1 > 0$
$\equiv \quad \forall x : x \ \epsilon \ \mathbb{Z} \rightarrow (x > 0 \rightarrow x + 1 > 0)$
$\equiv \quad \forall x : (x \ \epsilon \ \mathbb{Z} \wedge x > 0) \rightarrow x + 1 > 0$

$\forall x : P(x) \equiv$ to a very large AND

Example

$\forall x \ \epsilon \ \mathbb{N} : P(x)$
$P(0) \wedge P(1) \wedge P(2) \wedge \ldots$

Existential Quantifier

Definition

The *Existential quantifier*, $\exists$ (there exists), says that a statement must be true for at least one value of the variable.

Example

There is a student in CMSC 250

$\exists$ x ϵ P such that x is a student in CMSC 250

where P is a set of all people.

Existential Quantifier

Example

$\exists$x ϵ Z: $x = x^2$

Definition

Therefore, if Q(x) is a predicate and D the domain of x, an existential statement has the form $\exists$x ϵ D such that Q(x).

Negation and Quantifier

Definition

The following equivalencies hold:

$$\neg\forall x\colon \mathrm{P(x)} \equiv \exists \mathrm{x} : \neg P(x)$$
$$\neg\exists x\colon \mathrm{P(x)} \equiv \forall \mathrm{x} : \neg P(x)$$

Quantifier version of De Morgan's laws

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Quantifier version of De Morgan's laws

Example

- not all humans are mortal is equivalent to finding some human that is not mortal.
- No human is mortal is equivalent to showing that all humans are not mortal.

Translation Examples

Example

- All crows are black

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  $\forall x: \mathrm{Crow}(x) \rightarrow \mathrm{Black}(x)$

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- All crows are black

  $\forall x : \text{Crow}(x) \rightarrow \text{Black}(x)$

- Some cows are brown

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  $\exists x : \text{Cow}(x) \wedge \text{Brown}(x)$

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  $\exists x : \mathrm{Cow}(x) \wedge \mathrm{Brown}(x)$

- No cows are blue

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- Some cows are brown

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- No cows are blue

  $\neg\exists x\text{:}\ \text{Cow}(x) \wedge \text{Blue}(x)$

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- All that glitters is not gold

Translation Examples

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  $\exists \mathrm{x} : \mathrm{Cow}(\mathrm{x}) \wedge \mathrm{Brown}(\mathrm{x})$

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- All that glitters is not gold

  $\neg\forall x\text{:}\ \mathrm{Glitters}(\mathrm{x}) \rightarrow \mathrm{Gold}(\mathrm{x})$

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- No shirt, no service

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- No shirt, no service $\forall$x: $\neg$Shirt(x) $\rightarrow$ $\neg$Served(x)
- Every event has a cause

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- Every event has a cause $\forall$x$\exists$y: Causes(y,x)
- Every even number greater than 2 can be expressed as the sum of two primes.

Translation Examples

Examples

- No shirt, no service $\forall x\colon \neg \mathrm{Shirt}(x) \rightarrow \neg \mathrm{Served}(x)$
- Every event has a cause $\forall x \exists y\colon \mathrm{Causes}(y,x)$
- Every even number greater than 2 can be expressed as the sum of two primes.

  $\forall x : (\mathrm{Even}(x) \wedge x > 2 \rightarrow \exists p \exists q\colon \mathrm{Prime}(p) \wedge \mathrm{Prime}(q) \wedge (x = p + q))$