CMSC 250
Discrete Structures

Predicate Logic

Establishing Truth and Falsity

- To show $\exists$ statement is true:

  Find an example in the domain where it is true.
- To show $\exists$ statement is false:

  Show false for every member of the domain.
- To show $\forall$ statement is true:

  Show true for every member of the domain.
- To show $\forall$ statement is false:

  Find an example in the domain where it is false.

Domain Matters

Definition

Is the following true:

$$\forall x \exists y : y < x$$

Domain Matters

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Is the following true:

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If Domain is $\mathbb{N}$? (Naturals)

Domain Matters

Definition

Is the following true:

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If Domain is $\mathbb{N}$? (Naturals)

If Domain is $\mathbb{Z}$? (Integers)

Domain Matters

Definition

Is the following true:

$$\forall x \exists y : y < x$$

If Domain is $\mathbb{N}$? (Naturals)

If Domain is $\mathbb{Z}$? (Integers)

If Domain is $\mathbb{Q}$? (Rationals)

Domain Matters

Definition

Is the following true:

$$\forall x \exists y : y < x$$

If Domain is $\mathbb{N}$? (Naturals)

If Domain is $\mathbb{Z}$? (Integers)

If Domain is $\mathbb{Q}$? (Rationals)

If Domain is $\mathbb{Q}^{>0}$? (Rationals that are > 0)

Domain Matters

Definition

Is the following true:

$$\forall x \exists y : y < x$$

If Domain is $\mathbb{N}$? (Naturals)

If Domain is $\mathbb{Z}$? (Integers)

If Domain is $\mathbb{Q}$? (Rationals)

If Domain is $\mathbb{Q}^{>0}$? (Rationals that are > 0)

If Domain is $\mathbb{Q}^{>=0}$? (Rationals that are $>= 0$)

Domain Matters

Definition

Is the following true:

$$\forall x \exists y : y < x$$

If Domain is $\mathbb{N}$? (Naturals)

If Domain is $\mathbb{Z}$? (Integers)

If Domain is $\mathbb{Q}$? (Rationals)

If Domain is $\mathbb{Q}^{>0}$? (Rationals that are > 0)

If Domain is $\mathbb{Q}^{>=0}$? (Rationals that are $>= 0$)

If Domain is $\mathbb{R}$? (Reals)

Domain Matters

Definition

Is the following true:

$$\forall x \exists y : y < x$$

If Domain is $\mathbb{N}$? (Naturals)
If Domain is $\mathbb{Z}$? (Integers)
If Domain is $\mathbb{Q}$? (Rationals)
If Domain is $\mathbb{Q}^{>0}$? (Rationals that are > 0)
If Domain is $\mathbb{Q}^{>=0}$? (Rationals that are $>= 0$)
If Domain is $\mathbb{R}$? (Reals)
If Domain is $\mathbb{C}$? (Complex)

Negations

Example

Statement: All cats are furry

Negations

Example

Statement: All cats are furry

Negation: There is at least one cat that is not furry

Definition

The negation of a universal statement ("all are") is logically equivalent to an existential statement ("some are not" or "there is at least one that is not").

Negations

Example

Statement: Some snowflakes are the same

Negations

Example

Statement: Some snowflakes are the same

Negation: No snowflakes are the same

Definition

The negation of an existential statement ("some are") is logically equivalent to a universal statement ("none are" or "all are not").

Vacuous Universal Statements

Example

- A bowl sits on a table and next to it is a pile of five blue and five gray balls.

Vacuous Universal Statements

Example

- A bowl sits on a table and next to it is a pile of five blue and five gray balls.
- Place three blue balls and one gray ball in the bowl.

Vacuous Universal Statements

Example

- A bowl sits on a table and next to it is a pile of five blue and five gray balls.
- Place three blue balls and one gray ball in the bowl.
- Is the following statement true or false?

All the balls in the bowl are blue

Vacuous Universal Statements

Example

- A bowl sits on a table and next to it is a pile of five blue and five gray balls.
- Place three blue balls and one gray ball in the bowl.
- Is the following statement true or false?

  All the balls in the bowl are blue

- Empty the bowl and consider the following:

  All the balls in the bowl are blue.

  Is it true or false?

Vacuous Universal Statements

Example

- All the balls in the bowl are blue
- Is it true or false?
- This statement is false, if and only of, its negative is true.

  And its negation is

Vacuous Universal Statements

Example

- All the balls in the bowl are blue
- Is it true or false?
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  And its negation is
- There exists a ball in the bowl that is not blue

Vacuous Universal Statements

Example

- All the balls in the bowl are blue
- Is it true or false?
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  And its negation is
- There exists a ball in the bowl that is not blue
- But the bowl is empty and therefore the negation is false

Vacuous Universal Statements

Example

- All the balls in the bowl are blue
- Is it true or false?
- This statement is false, if and only of, its negative is true.
  And its negation is
- There exists a ball in the bowl that is not blue
- But the bowl is empty and therefore the negation is false
- that makes the original statement true by "default"

Vacuous Universal Statements

Example

- If domain, D, is empty, then $(\exists x \epsilon D)[P(x)]$ is vacuously falls.
- If domain, D, is empty, then $(\forall x \epsilon D)[P(x)]$ is vacuously true.

Does order matter

Order

Let P(x,y) be a statement, x + y = y + x

if x,y $\epsilon\mathbb{R}$

Are,

$$\forall x \forall y \; P(x,y), \text{ and}$$
$$\forall y \forall x \; P(x,y)$$

equivalent?

Does order matter

Order

Let Q(x,y) be a statement, $x + y = 0$

if x,y $\epsilon \mathbb{R}$

Are,

$$\exists y \forall x \ Q(x,y), \text{ and}$$
$$\forall x \exists y \ Q(x,y)$$

equivalent?

Does order matter

Example

Q(x,y,z): $x + y = z$, for x,y,z ϵ $\mathbb{R}$

Are the statements,

$$\forall x \forall y \exists z \; Q(x,y,z)$$
$$\exists z \forall x \forall y \; Q(x,y,z)$$

equivalent?

Nested Quantifiers

Example

Translate the statement

$\forall x(C(x) \vee \exists y(C(y) \wedge F(x,y)))$

where,
C(x): "x has a computer"
F(x,y): "x and y are friends"
and the domain for x and y consists of all students in your school.

Translate

Example

If a person is a female and is a parent, then this person is a mother

Translate

Example

If a person is a female and is a parent, then this person is a mother

For every person, x, if person x is female, and person x is a parent, then there exists a person y such that person x is the mother of person y.
x,y ϵ all the people

Translate

Example

If a person is a female and is a parent, then this person is a mother

For every person, x, if person x is female, and person x is a parent, then there exists a person y such that person x is the mother of person y.
x,y ϵ all the people

F(x): x is a female
P(x): x is a parent
M(x,y): x is the mother of y

Translate

Example

If a person is a female and is a parent, then this person is a mother

For every person, x, if person x is female, and person x is a parent, then there exists a person y such that person x is the mother of person y.
x,y ϵ all the people

F(x): x is a female
P(x): x is a parent
M(x,y): x is the mother of y

$\forall x \exists y (F(x) \wedge P(x) \rightarrow M(x,y)))$

Translate

Example

Everyone has exactly one best friend

Translate

Example

Everyone has exactly one best friend

For every person, x, person x has exactly one best friend

Translate

Example

Everyone has exactly one best friend

For every person, x, person x has exactly one best friend

B(y,x): y is the best friend of x

Translate

Example

Everyone has exactly one best friend

For every person, x, person x has exactly one best friend

B(y,x): y is the best friend of x

$\forall x \exists y (B(y,x) \wedge \forall z((z \neq y) \rightarrow \neg B(z,x)))$

x,y,z ϵ all the people

Translate

Example

There is a man who has taken a flight on every airline in the world.

Translate

Example

There is a man who has taken a flight on every airline in the world.

P(m,f): m has taken f
Q(f.a): f is a flight on a
m ϵ all women in the world, f ϵ all airplane flights, and, a ϵ all airplanes.

Translate

Example

There is a man who has taken a flight on every airline in the world.

P(m,f): m has taken f
Q(f.a): f is a flight on a
m ϵ all women in the world, f ϵ all airplane flights, and, a ϵ all airplanes.

$\exists m \forall a \exists f (P(m,f) \wedge Q(f,a))$

Negating Nested Quantifiers

Definition

Express negation of the statement $\forall x \exists y (xy = 1)$

Example

Recall

The following equivalencies hold:

$$\neg \forall x\colon \mathrm{P(x)} \equiv \exists \mathrm{x} : \neg P(x)$$
$$\neg \exists x\colon \mathrm{P(x)} \equiv \forall \mathrm{x} : \neg P(x)$$

Quantifier version of De Morgan's laws

Negating Nested Quantifiers

Definition

Express negation of the statement $\forall x \exists y (xy = 1)$

Solution

$$\begin{aligned} \neg \forall x \exists y (xy = 1) &\equiv \exists x \neg \exists y (xy = 1) \\ &\equiv \exists x \forall y \neg (xy = 1) \\ &\equiv \exists x \forall y (xy \neq 1) \end{aligned}$$