CMSC 250
Discrete Structures
Induction
Example

- Let $P(n)$ be the sentence “$n$ cents postage can be obtained using 3¢ and 5¢ stamps”.
- Want to show that “$P(k)$ is true” implies “$P(k+1)$ is true” for all $k \geq 8$.
- 2 cases:
  1) $P(k)$ is true and the $k$ cents contain at least one 5¢.
  2) $P(k)$ is true and the $k$ cents do not contain any 5¢.
Example

Case 1: \( k \) cents contain at least one 5¢ stamp.

\[ \begin{array}{c}
\text{k cents} \\
\text{5¢} \\
\end{array} \] \quad \text{Replace 5¢ stamp by two 3¢ stamps} \quad \begin{array}{c}
\text{k+1 cents} \\
\text{3¢ 3¢} \\
\end{array} \]

Case 2: \( k \) cents do not contain any 5¢ stamp. Then there are at least three 3¢ stamp.

\[ \begin{array}{c}
\text{k cents} \\
\text{3¢ 3¢ 3¢} \\
\end{array} \] \quad \text{Replace three 3¢ stamps by two 5¢ stamps} \quad \begin{array}{c}
\text{k+1 cents} \\
\text{5¢ 5¢} \\
\end{array} \]
1. I know that the FIRST domino falls (because I am knocking it over).
2. I can prove that if any particular domino falls, then the very next one must also fall.

What can I conclude?
Simple induction

**Definition**

Claim: $(\forall n \in \mathbb{N})[P(n)]$

Proof:

I will induct on $n$

Base case: Show $P(0)$ directly.

Inductive hypothesis: Assume, $P(k)$ is true, for some $k \in \mathbb{N}$.

Inductive Step: Prove $P(k+1)$ must also be true based on your assumption that $P(k)$ is true.
Definition

Claim: \((\forall n \in \mathbb{N}^>0)[\text{Domino} \ #n \ \text{will fall}]\)
Proof: By induction on \(n\)
Base case: Domino \#1 will fall because I push it over.
Inductive hypothesis: Assume domino \(k\) falls for some \(k \in \mathbb{N}^>0\)
Inductive Step: Since domino \(k\) is falling, it will strike domino \(k + 1\), knocking it over (‘cause that is how Physics works’).
Recall the Modular Arithmetic Theorem:

<table>
<thead>
<tr>
<th>Definition</th>
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<tbody>
<tr>
<td>Let $a, b, c, d, n \in \mathbb{Z}$, and $n &gt; 1$. Suppose $a \equiv_n c$ and $b \equiv_n d$. Then:</td>
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<tr>
<td>1. $(a+b) \equiv_n (c + d)$</td>
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<td>2. $(a-b) \equiv_n (c - d)$</td>
</tr>
<tr>
<td>3. $ab \equiv_n cd$</td>
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<tr>
<td>4. $a^m \equiv_n c^m$ for all natural number $m$</td>
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Another Example

Example

$$(\forall n \in \mathbb{N}^{\geq 1})[n^3 \equiv_3 n]$$
**Examples with Summations**

<table>
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<tr>
<td><strong>Claim:</strong> $(\forall n \geq 1) \left[ \sum_{i=1}^{n} 4i - 2 = 2n^2 \right]$</td>
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<tr>
<td><strong>Claim:</strong> $(\forall n \geq 1) \left[ \sum_{i=1}^{n} i = \frac{n(n + 1)}{2} \right]$</td>
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<tr>
<td><strong>Claim:</strong> $(\forall n \geq 0) \left[ \sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \right]$</td>
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Another Example - geometric series

Example

$$(\forall r \in \mathbb{R}^>^1)(\forall n \in \mathbb{Z}^\geq^0)\left[\sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1}\right]$$
An Example - with an inequality

Example

Claim:

\[(\forall n \in \mathbb{Z}_{\geq 3})[2n + 1 < 2^n]\]