CMSC 250
Discrete Structures

Induction

Example

- Let P(n) be the sentence "n cents postage can be obtained using 3¢ and 5¢ stamps".
- Want to show that

  "P(k) is true" *implies* "P(k+1) is true"

  for all $k \geq 8$.
- 2 cases:

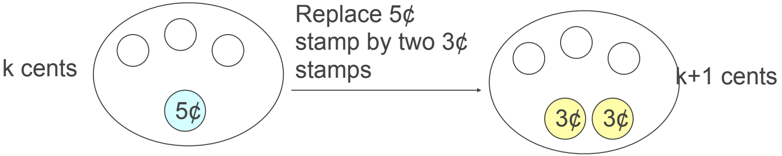
1) P(k) is true and

   the k cents contain at least one 5¢.
2) P(k) is true and

   the k cents do not contain any 5¢.

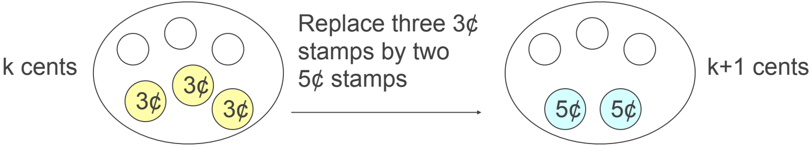
Example

Case 1: k cents contain at least one 5¢ stamp.



Case 2: k cents do not contain any 5¢ stamp.

Then there are at least three 3¢ stamp.



Basic Idea

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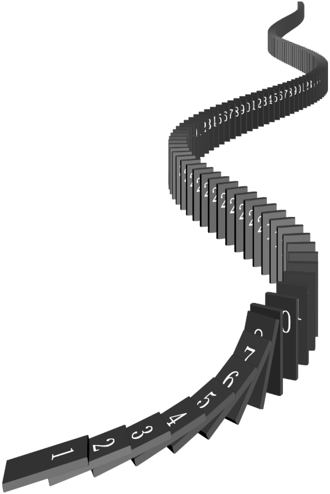

1. I know that the FIRST domino falls (because I am knocking it over).
2. I can prove that if any particular domino falls, then the very next one must also fall.

What can I conclude?

Simple induction

Definition

Claim: $(\forall n \in \mathbb{N})[P(n)]$

Proof:

I will induct on n

Base case: Show P(0) directly.

Inductive hypothesis: Assume, P(k) is true, for some $k \in \mathbb{N}$.

Inductive Step: Prove P(k+1) must also be true based on your assumption that P(k) is true.

Domino "Proof"

Definition

Claim: $(\forall n \in \mathbb{N}^{>0})$[Domino #n will fall]
Proof: By induction on n
Base case: Domino #1 will fall because I push it over.
Inductive hypothesis: Assume domino k falls for some $k \in \mathbb{N}^{>0}$
Inductive Step: Since domino k is falling, it will strike domino $k + 1$, knocking it over ('cause that is how Physics works').

Simple Example

Recall the Modular Arithmetic Theorem:

Definition

Let a, b, c, c, n $\in \mathbb{Z}$, and $n > 1$. Suppose $a \equiv_n c$ and $b \equiv_n d$. Then:

1. (a+b) $\equiv_n$ (c + d)
2. (a-b) $\equiv_n$ (c - d)
3. ab $\equiv_n$ cd
4. $a^m \equiv_n c^m$ for all natural number m

Another Example

Example

$(\forall n \in \mathbb{N}^{\geq 1})[n^3 \equiv_3 n]$

Examples with Summations

Example

- Claim:
$$(\forall n \geq 1)\left[\sum_{i=1}^{n} 4i - 2 = 2n^2\right]$$
- Claim:
$$(\forall n \geq 1)\left[\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\right]$$
- Claim:
$$(\forall n \geq 0)\left[\sum_{i=0}^{n} 2^i = 2^{n+1} - 1\right]$$

Another Example - geometric series

Example

$$(\forall r \in \mathbb{R}^{>1})(\forall n \in \mathbb{Z}^{\geq 0})\left[\sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1}\right]$$

An Example - with an inequality

Example

Claim:

$$(\forall n \in \mathbb{Z}^{\geq 3})\left[2n + 1 < 2^n\right]$$