CMSC 250
Discrete Structures

Induction

An Example - with an inequality

Example

Claim:

$$(\forall n \in \mathbb{Z}^{\geq 3})\left[2n + 1 < 2^n\right]$$

Another Example with an inequality

Example

$$(\forall n \in \mathbb{Z}^{\geq 2})(\forall x \in \mathbb{R}^{+})\left[1 + nx \leq (1 + x)^n\right]$$

An example with a recurrence relation

- Assume the following definition of a sequence:

$$a_1 = 1$$

$$(\forall k \geq 2)\left[a_k = a_{k-1} + (2k - 1)\right]$$

- Prove: $(\forall n \geq 1)\left[a_n = n^2\right]$

Recurrence relation and summation together

- Assume the following definition of a sequence:
  $a_0 = 1$

$$\text{For} n \geq 1 : a_n = \left[\sum_{i=0}^{n-1} a_i \right] + 1$$

- Prove: $(\forall n \geq 0) \left[a_n = 2^n \right]$

Another recurrence to consider

- Assume the following definition of a function:
  $a_0 = 1 \qquad a_1 = 1 \qquad a_2 = 3$
  $(\forall k \in Z^{\geq 3})[a_k = a_{k-1} + a_{k-2} - a_{k-3}]$

Claim: $(\forall n \in \mathbb{Z}^{\geq 0})[a_n \in \mathbb{Z}^{odd}]$

Can we prove this with induction?

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What happens if we knock the first three dominos off the table manually?

Strong induction

For (ordinary) induction:

- We assume one particular domino falls and then show that the next one must also fall.

For strong induction:

- We assume all of the dominos before domino $k + 1$ fall and then show domino $k + 1$ must also fall.

This is frequently easier because our inductive hypothesis is stronger (we are assuming more stuff).

Strong induction

Definition

Claim: $(\forall n \in \mathbb{N})[P(n)]$

Proof:

I will apply strong induction on n

Base case: Show P(0), P(1), P(2), . . ., P(7) directly.

Inductive hypothesis: For some $k \geq 7$: Assume P(i) holds for all $i \leq k$.

Inductive Step: Prove P(k+1) must also be true, based on your assumption that P holds for all previous values.

Why Does Strong Induction Work?

$P(1)$
$(\forall n \geq 2)[P(1) \wedge P(2) \wedge \cdots \wedge P(n-1) \rightarrow P(n)]$

1) Have $P(1)$
2) Have $P(1) \rightarrow P(2)$
3) From $P(1)$ and $P(1) \rightarrow P(2)$ have $P(2)$.
4) Have $P(1) \wedge P(2) \rightarrow P(3)$.
5) From $P(1), P(2))$ and $P(1) \wedge P(2) \rightarrow P(3)$ have $P(3)$.
etc.

Now prove the recurrence relation property, using strong induction

- Here is the function definition again:
  $a_0 = 1 \qquad a_1 = 1 \qquad a_2 = 3$
  $(\forall k \in Z^{\geq 3})[a_k = a_{k-1} + a_{k-2} - a_{k-3}]$

Claim: $(\forall n \in \mathbb{Z}^{\geq 0})[a_n \in \mathbb{Z}^{odd}]$

Another recurrence relation example

- Assume the following definition of a function:

$$a_0 = 1 \qquad a_1 = 2$$

$$(\forall k \in Z^{\geq 2})\left[a_k = a_{k-1} + a_{k-2}\right]$$

- Prove the following definition property, using strong induction:

$$(\forall n \in Z^{\geq 0})\left[a_n \leq 2^n\right]$$

Another example- a divisibility property

- Assume the following definition of a recurrence relation:

$$a_0 = 0$$

$$a_1 = 7$$

$$(\forall i \geq 2)\left[a_i = 2a_{i-1} + 3a_{i-2}\right]$$

- Prove using strong induction that all elements in this relation have this property:

$$(\forall n \in N)\left[a_n \equiv 0 \pmod 7\right]$$