① Proof: by Cases

There are two cases we will look at

Case (i) $n \leq 4$, Case (ii) ≥ 5

Case (i) n^3 when $n = 4$

$4^3 = 64 < 100$

Case (ii) $n = 5$

$n^3 = 5^3 = 125 > 100$

There exists no positive integer whose cube is 100.

In Case (i) 4 is the largest number, any +ve integer less 4 will be < 100

Similarly in Case (ii) 5 is the smallest number, any number larger 5 will have a cube larger than 100.

① Constructive Proof of Existence

Proof:

$3^2 = 9$ — is a perfect square

$2^3 = 8$ — is a perfect cube

8 and 9 are consecutive integers

Congruent Modulus Theorem

$$a \equiv b \mod m$$

This is equivalent to

$m \mid (a-b)$ (m divides $a-b$)

By quotient remainder Theorem

$a - b = km$ where $k \in \mathbb{Z}$

Therefore, $a = b + km$

$$a + c = b + d + (k + q)m$$

$$(a + c) = (b + d) + pm$$

where $p \in \mathbb{Z}$ and

$$k + q = p$$

$p \in \mathbb{Z}$ because integers are closed in addition

$$(a + c) = (b + d) + pm$$

$$(a + c) - (b + d) = pm$$

$$\Rightarrow \quad m \mid (a + c) - (b + d)$$

$$\Rightarrow \quad a + c \equiv (b + d) \bmod m$$

We have shown what was to be proved

(4) $a \equiv b \mod m$

$c \equiv d \mod m$

Show $a+c = b+d \pmod m$

an $ac = bd \pmod m$

<u>Proof:</u> for $a+c = b+d \mod m$

$a \equiv b \mod m$

$km = a-b \quad (\because m \mid a-b)$

where $k \in \mathbb{Z}$

$a - b = km \Rightarrow a = b+km$ — (1)

Similarly, since $c \equiv d \mod m$

$c - d = qm \quad (\because m \mid c-d)$

where $q \in \mathbb{Z}$

$c = d + qm$ — (2)

Adding eq.'s (1) and 2

Show, $ac = bd \pmod{m}$

when $a \equiv b \bmod m$

$c \equiv d \bmod m$

$a = b + km$ (~~Rem~~ Quotient remainder theorem)

$c = d + qm$ (Again by quotient remainder)

Multiply, a to c

$$ac = (b+km)(d+qm)$$

$$ac = bd + bqm + kdm + kqm^2$$

$$ac = bd + m(bq + kd + kqm)$$

$\Rightarrow$ m divides $ac - bd$

since $(bq + kd + kqm) \in \mathbb{Z}$

because integers are closed in multiplication and addition.

We have shown what was to be proved.

$$4^{\boxed{2349321230}}$$ when divided by 15

~~Proof~~ ~~Pro~~ Solution

Using Congruent modulo theorem

$$4^1 \equiv 4 \mod 15$$

$$4^2 \equiv 1 \mod 15$$

$$4^3 \equiv 4 \mod 15$$

$$4^4 \equiv 1 \mod 15$$

When the exponent is even, the remainder is 1 and when the exponent is odd, the remainder is 4

This pattern is repeating. ~~so~~,

The exponent of 4 is 2349321230 is an even number,

Therefore, the remainder of $4^{2349321230}$ mod m is 1

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2}. & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof: by Cases

~~Case~~

Case i, n is even

By definition $\underline{\underline{n = 2k}}$, where $k \in \mathbb{Z}$

Thus, $\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2k}{2} \right\rfloor = \lfloor k \rfloor = k$

$$\Rightarrow \quad k = \frac{n}{2}$$

because the floor of an integer is the integer itself.

Case (ii) n is odd

By definition $n = 2k+1$
where $k \in \mathbb{Z}$

Thus, $\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2k+1}{2} \right\rfloor = \left\lfloor k + \frac{1}{2} \right\rfloor = k.$

The floor of an integer plus a real value is the integer before the sum.

since $n = 2k+1$

$n - 1 = 2k$

$k = \frac{n-1}{2}$

$$\sum_{i=50}^{100} i = \sum_{i=1}^{100} i - \sum_{i=1}^{49} i$$

$$= \frac{100(101)}{2} - \frac{49(50)}{2}$$

49×25

$$= 5050 - \frac{2450}{2}$$

$$= 5050 - 1225$$

Proof by: Mathematical Induction.

Base Case : $n=1$, <u>Show for $n=1$</u>

<u>I. H.</u> : Assume <u>it</u> → theorem holds true for n

e.g. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

<u>Inductive Step</u> : Show for $n+1$

Strong Induction

Base Case: $n = 1$

I. hypothesis: Assume it holds true for all values $\leq n$

i.e for any arbitrary $k = 1, 2, 3, \ldots n$.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{or} \quad \sum_{i=1}^{n-1} i = \frac{(n-1)(n)}{2}$$

or $$\sum_{i=1}^{n-2} i = \frac{(n-2)(n-1)}{2} \ldots$$

<u>Inductive Step</u>: Show for $n+1$

Constructive Induction

Start with a guess solution

Base Case: $n = 1$

Inductive hypothesis: Assume it holds for all values $\leq n$ (write formula for the last value)

Inductive Step: Show for $n+1$

Show where you're applying I.H.

for $n \geq 1$

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + n \cdot n! = (n+1)! - 1$$

Proof: By induction on n

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + n \cdot n! = \sum_{i=1}^{n} i \cdot i!$$

$$\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1$$

Base Case: $n = 1$

$$\sum_{i=1}^{1} i \cdot i! = 1 \cdot 1! = 1$$

$$(n+1)! - 1 = (1+1)! - 1 = 2 - 1 = 1$$

It holds for the base case.

<u>Inductive hypothesis</u>: Assume it holds true for n,

$$\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1$$

Inductive Step: Show for $n+1$

$$\sum_{i=1}^{n+1} i \cdot i! = (n+2)! - 1$$

$$\sum_{i=1}^{n+1} i \cdot i! = \sum_{i=1}^{n} i \cdot i! + (n+1)(n+1)!$$

$$= (n+1)! - 1 + (n+1)(n+1)!$$

[By I.H.]

$$= (n+1)! - 1 + (n+1)(n+1)!$$

$$= (n+1)! \left[1 + n+1\right] - 1$$

$$= (n+1)! \left[n+2\right] - 1$$

$$= (n+2)! - 1$$

We have shown what was to be proved

$a_1 = 1, \; a_2 = 8$

$a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$

Show $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in \mathbb{N}$

Proof: By Strong induction on n

Base Case: $n = 1$, $a_1 = 1$

$$a_1 = 3 \cdot 2^{1-1} + 2(-1)^1$$
$$= 3 - 2 = 1$$

$\underline{a_2 = 8}$

$$\underline{a_2} = 3 \cdot 2^{2-1} + 2(-1)^2$$
$$= 3 \cdot 2 + 2$$
$$= \underline{8}$$

This holds true for the base cases.

<u>I.H.</u>: Assume it holds true for n, $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$

and $a_{n-1} = 3 \cdot 2^{n-2} + 2(-1)^{n-1}$

I. Step.: Show $a_{n+1} = \underline{\underline{3 \cdot 2^n}} + 2(-1)^{n+1}$

$$a_{n+1} = a_n + 2a_{n-1} \quad \text{[By definition]}$$

$$= 3 \cdot 2^{n-1} + 2(-1)^n + 2\left[3 \cdot 2^{n-2} + 2(-1)^{n-1}\right]$$

[By I.H.]

$$= 3 \cdot 2^{n-1} + 2(-1)^n + 3 \cdot 2 \cdot 2^{n-2} + 2 \cdot 2(-1)^{n-1}$$

$$3 \cdot 2^{n-1}\cancel{[1+1]} = \underline{3 \cdot 2^{n-1} + 3 \cdot 2^{n-1}} + 2(-1)^n + 2 \cdot 2(-1)^{n-1}$$

$$= 2 \cdot 3 \cdot 2^{n-1} + \underline{2}(-1)^n + 2 \cdot 2(-1)^{n-1}$$

$$= 3 \cdot 2^n + 2(-1)^{n+1}[\quad\quad]$$

$$a_{n+1} = 3\cdot 2^{n} + 2(-1)^{n+1}\left[(-1)^{-1} + 2\cdot(-1)^{-2}\right]$$

$$= 3\cdot 2^{n} + 2(-1)^{n+1}\left[\frac{1}{-1} + 2\cdot\frac{1}{(-1)^{2}}\right]$$

$$= 3\cdot 2^{n} + 2(-1)^{n+1}\left[-1 + 2\right]$$

$$= 3\cdot 2^{n} + 2(-1)^{n+1}\cdot 1$$

$$= 3\cdot 2^{n} + 2(-1)^{n+1}$$

$$\underbrace{3\cdot 2^{n-1}}_{a} + \underbrace{3\cdot 2^{n-1}}_{a}$$

$$3\cdot 2^{n-1}\left[1 + 1\right]$$

$$\sum_{i=1}^{n} i^2$$

Proof: By Constructive induction on n

Guess: We know $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$$\sum_{i=1}^{n} i^2 = an^3 + bn^2 + cn + d$$

Base case: $n = 1$

$$\sum_{i=1}^{1} i^2 = 1^2 = 1$$

This should match our guess

$$an^3 + bn^2 + cn + d = a + b + c + d$$

$$a + b + c + d = 1 \quad \text{--- (1)}$$

I.H.: It holds true for all values $\leq n - 1$

$$\sum_{i=1}^{n-1} i^2 = a(n-1)^3 + b(n-1)^2 + c(n-1) + d$$

I. Step: Show for n

$$\sum_{i=1}^{n} i^2 = an^3 + bn^2 + cn + d$$

$$\sum_{i=1}^{n} i^2 = \underline{\underline{\sum_{i=1}^{n-1} i^2}} + n^2$$

$$= a(n-1)^3 + b(n-1)^2 + c(n-1) + d + n^2$$

[By. I.H]

$$= a(n^3 - 3n^2 + 3n - 1) + b(n^2 - 2n + 1) + cn - c + d + n^2$$

$$= \underline{an^3} - 3\underline{an^2} + \underline{\underline{3an}} - a + \underline{bn^2} - \underline{\underline{2bn}} + b + \underline{\underline{c}}\,\underline{n} - c + d + \underline{n^2}$$

$$= an^3 + bn^2 + cn + d$$

$$-3a + b + 1 = b \quad -(2)$$

$$3a - 2b + c = c \quad -(3)$$

$$-a + b - c + d = d \quad -(4)$$

$$-3a+b+1 = b \quad — (2)$$

$$3a - 2b + c = c \quad — (3)$$

$$-a + b - c + d = d \quad — (4)$$

$$a+b+c+d = 1 \quad — (1)$$

(from base case)

From eq. (2)

$$-3a+b+1 = b$$

$$\Rightarrow \quad -3a = -1 \Rightarrow a = 1/3$$

From eq (3)

$$3a - 2b + c = c$$

$$\Rightarrow \quad 3a - 2b = 0$$

$$a = 1/3 \Rightarrow 3 \cdot \frac{1}{3} - 2b = 0$$

$$b = 1/2$$

From eq. (4) $-a+b-c+d = d$

$$\Rightarrow \quad -a+b = c$$

$$\Rightarrow \quad -1/3 + 1/2 = c \Rightarrow c = \frac{1}{6}$$

From base case: $a+b+c+d = 1$

$$\frac{1}{3} + \frac{1}{2} + \frac{1}{6} + d = 1 \Rightarrow d = 0$$

Therefore,

$$\sum_{i=1}^{n} i^2 = an^3 + bn^2 + cn + d$$

$$= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} + 0$$

$$= \frac{n}{6}\left(2n^2 + 3n + 1\right)$$

$$= \frac{n\left(2n^2 + 2n + n + 1\right)}{6}$$

$$= \frac{n\left(2n(n+1) + 1(n+1)\right)}{6}$$

$$= \frac{n(2n+1)(n+1)}{6}$$