CMSC 250
Midterm II

Review

Proof by cases

Example

Use a proof by cases to show that 100 is not the cube of a positive integer.

Constructive Proofs of Existence

Prove that there exists a pair of consecutive integers such that one of these integers is a perfect square and the other is a perfect cube.

Universal Generalization - Method of Proof

- The most common technique for proving universally quantified statements.
- If you're not sure how to start - try this!

Definition

Theorem: $(\forall x \in D)[P(x)]$
Proof:
let $a \in D$, arbitrarily chosen.
$\ldots$
P(a)
Since a was chosen arbitrarily, $P(x)$ holds for all $x \in D$.

Congruent Modulo Theorem

Let m be a positive integer. The integers a and b are congruent modulo m if and only if there is an integer k such that $a = b + km$

Congruent Modulus properties

Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then
$a + c \equiv b + d \pmod{m}$ and $ac \equiv bd \pmod{m}$

Fundamental Theorem of Arithmetic (Unique Prime Factorization Theorem)

Definition

Given any integer $n > 1$, there exist a positive integer k, distinct prime numbers $p_1, p_2, \ldots, p_k$, and positive integers $e_1, e_2, \ldots, e_k$ such that

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \ldots p_k^{e_k}$$

and $p_1 < p_2 < \ldots < p_k$.

Quotient Remainder Theorem

Definition

Given any integer n and positive integer d, there exist unique integers q and r such that

$$n = dq + r \text{ and } 0 \leq r < d$$

Quotient Remainder Theorem Representation

If we represent integers using the quotient remainder theorem, we observe

Modulus	Forms
2	$2q$, $2q + 1$
3	$3q$, $3q + 1$, $3q + 2$
4	$4q$, $4q + 1$, $4q + 2$, $4q + 3$
....	
k	kq, $kq + 1$, $kq + 2$ ... $kq + (k\text{-}1)$

Remainder

Find remainder of $4^{2349321230}$ when divided by 15.

Floors and Ceilings

For any integer n,

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Summation

$$\sum_{i=50}^{100} i$$

Mathematical Induction

For $n \geq 1$

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n! = (n+1)! - 1$$

Strong Induction

$a_1 = 1, a_2 = 8$ and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$
Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all n $\in \mathbb{N}$

Constructive Induction

Find

$$\sum_{i=1}^{n} i^2$$