CMSC 250
Discrete Structures

Set Theory

Set Definitions

Definition

A *set* is an unordered collection of elements.
name of set $=$ {list of elements, or a description of the elements}

Example

S $= \{$a,b,c,d$\}$, $a \in S$ and $e \notin S$
A $= \{1,2,3\}$
B $= \{x \in Z | -4 < x < 4\}$
C $= \{x \in Z^{+} | -4 < x < 4\}$

More Set Concepts

Definition

- The universal set, U, is the set consisting of all possible elements in some particular situation under consideration.
- A set can be finite or can be infinite.
- For a set S, n(S) or $|S|$ are used to refer to the cardinality of S, which is the number of elements in S.

Subset

Definition

- $A \subseteq B \leftrightarrow (\forall x \in U)[x \in A \rightarrow x \in B]$
  A is contained in B
  B contains A
- $A \not\subseteq B \leftrightarrow (\exists x \in U)[x \in A \wedge x \notin B]$
- Relationship between membership and subset:
  $(\forall x \in U)[x \in A \leftrightarrow \{x\} \subseteq A]$
- Definition of set equality: $A = B \leftrightarrow A \subseteq B \wedge B \subseteq A$

Formal definitions of set operations

Definition

Union: $A \cup B = \{x \in U | x \in A \vee x \in B\}$

Intersection: $A \cap B = \{x \in U | x \in A \wedge x \in B\}$

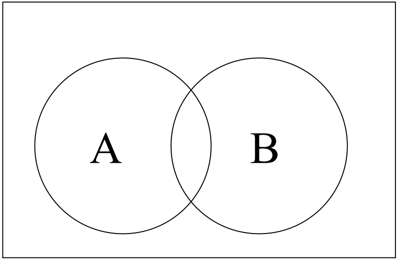
Complement: $A^c = A' = \bar{A} = \{x \in U | x \notin A\}$

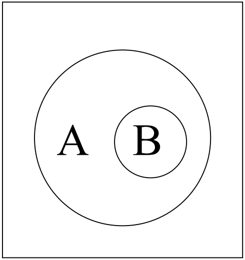
Difference: $A - B = \{x \in U | x \in A \wedge x \notin B\}$

$$A - B = A \cap B'$$

Venn diagrams

Sets are represented as regions (usually circles) in the plane in order to graphically illustrate relationships between them.





- Practice identifying union, intersection, difference compliment
- Can we draw Venn diagrams with more than 2 sets?

The empty set and its properties

The empty set $\emptyset$ has no elements, so $\emptyset = \{\}$.

1. $(\forall \text{ sets } X)[\emptyset \subseteq X]$ (Why?)
2. There is only one empty set. (Why?)
3. $(\forall \text{ sets } X)[X \cup \emptyset = X]$
4. $(\forall \text{ sets } X)[X \cap X' = \emptyset]$
5. $(\forall \text{ sets } X)[X \cap \emptyset = \emptyset]$
6. $U' = \emptyset$
7. $\emptyset' = U$

Ordered n-tuples

- An ordered n-tuple takes order and multiplicity into account
- The tuple $(x_1,x_2,x_3,\ldots,x_n)$
  - has n values
  - which are not necessarily distinct
  - and which appear in the order listed
- $(x_1,x_2,x_3,\ldots,x_n) = (y_1,y_2,y_3,\ldots,y_n) \leftrightarrow (\forall i \in 1 \leq i \leq n)[x_i = y_i]$
- 2-tuples are called pairs, and 3-tuples are called triples

The Cartesian product

- The Cartesian product of sets *A* and *B* is defined as

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

- $n(A \times B) = n(A) * n(B)$

Proper subset

$$A \subset B \leftrightarrow A \subseteq B \wedge A \neq B$$

Disjoint sets

A and B are disjoint

$\leftrightarrow$ A and B have no elements in common

$\leftrightarrow (\forall x \in U)[x \in A \rightarrow x \notin B \ \wedge \ x \in B \rightarrow x \notin A]$

$A \cap B = \varnothing \ \leftrightarrow$ A and B are disjoint sets

Power set

$\mathcal{P}$(A) = the set of **all** subsets of A

Examples- what are $\mathcal{P}(\{a\})$?

$\mathcal{P}(\{a,b,c\})$?

$\mathcal{P}(\emptyset)$?

$\mathcal{P}(\{\emptyset\})$?

$\mathcal{P}(\{\emptyset,\{\emptyset\}\})$?

Some Properties of Sets

- Inclusion $A \bigcap B \subseteq A$ $A \bigcap B \subseteq B$

  $A \subseteq A \bigcup B$ $B \subseteq A \bigcup B$

- Transitivity $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$

- Let's prove a few of these

Proving two sets are equal

Two (basic) techniques:

Claim: A = B.
Proof:

$x \in A \leftrightarrow$

$S1 \leftrightarrow$

$S2 \leftrightarrow$

$S3 \leftrightarrow$

…

$x \in B$

Claim: A = B.
Proof:

Part I. [Show $A \subseteq B$]

…

Part II. [Show $B \subseteq A$]

…

More Properties of Sets

- DeMorgan's for complement

$$(A \bigcup B)' = A' \bigcap B'$$

$$(A \bigcap B)' = A' \bigcup B'$$

- Distribution of union and intersection

$$A \bigcup (B \bigcap C) = (A \bigcup B) \bigcap (A \bigcup C)$$

$$A \bigcap (B \bigcup C) = (A \bigcap B) \bigcup (A \bigcap C)$$

- Let's prove a couple.
- There are a number of others as well; see the handout that is on the class webpage