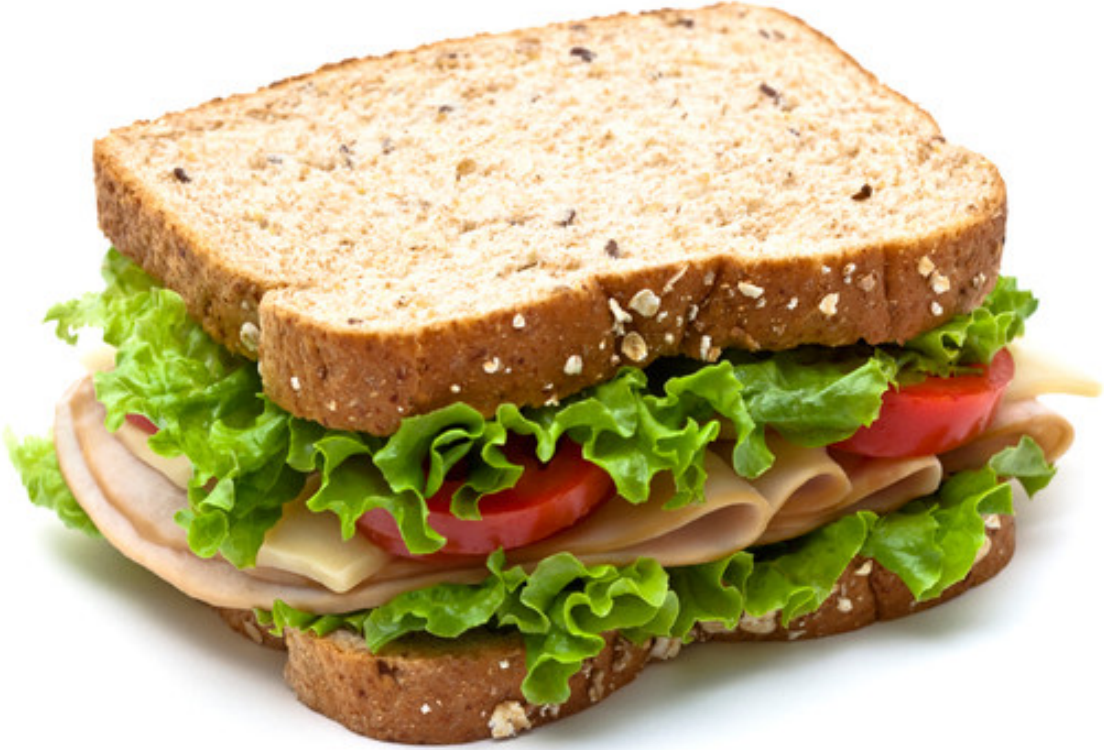
Intro to Combinatorics
(“that n choose 2 stuff”)

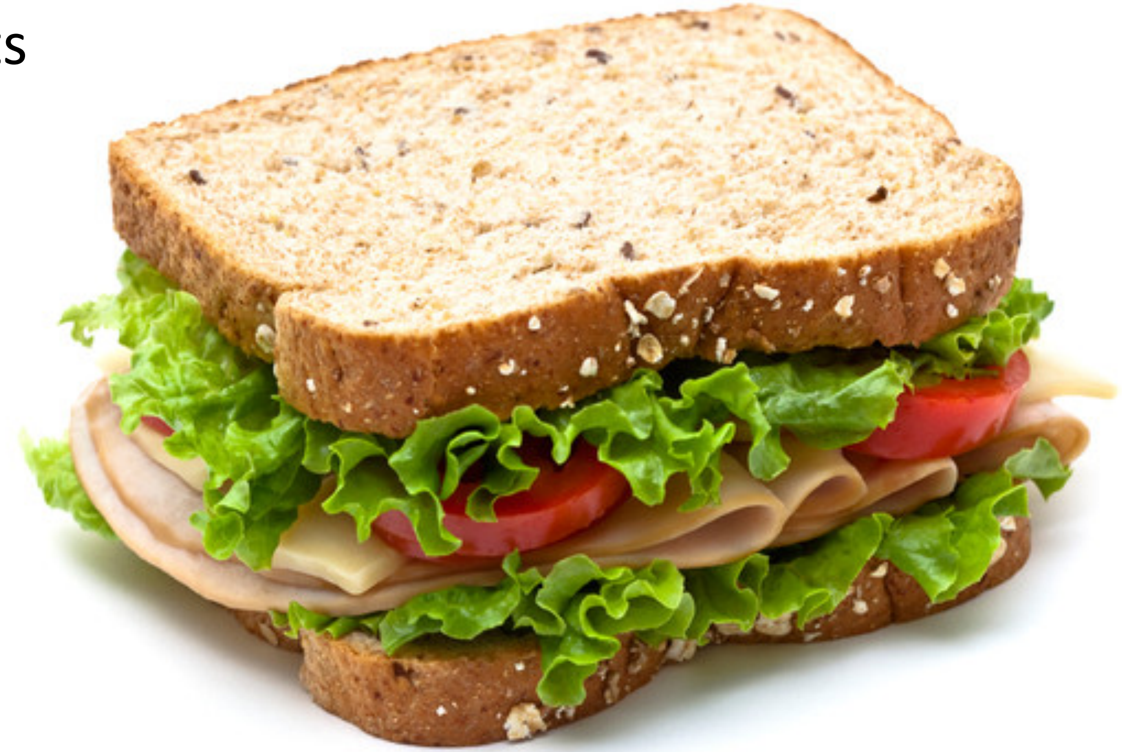
CMSC 250

Jason’s sandwich



Jason's sandwich

- Suppose that Jason has the following ingredients to make a sandwich with:
  - White or black bread
  - Butter, Mayo or Honey Mustard
  - Romaine Lettuce, Spinach, Kale
  - Bologna, Ham or Turkey
  - Tomato or egg slices



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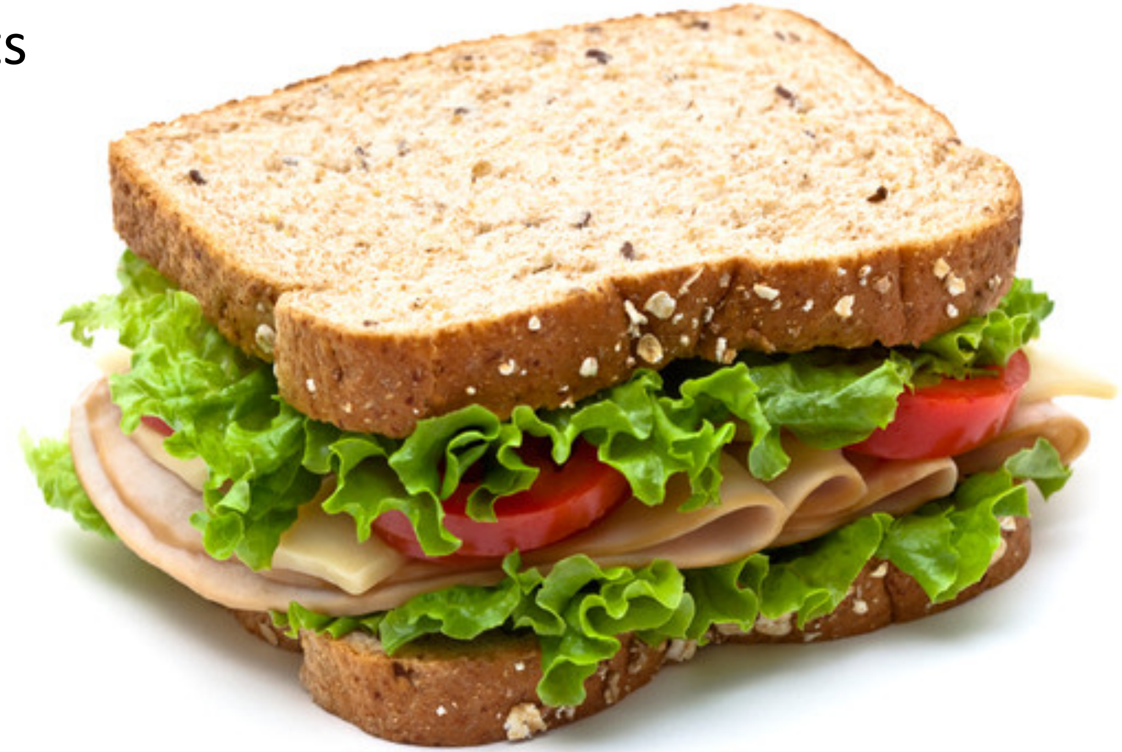
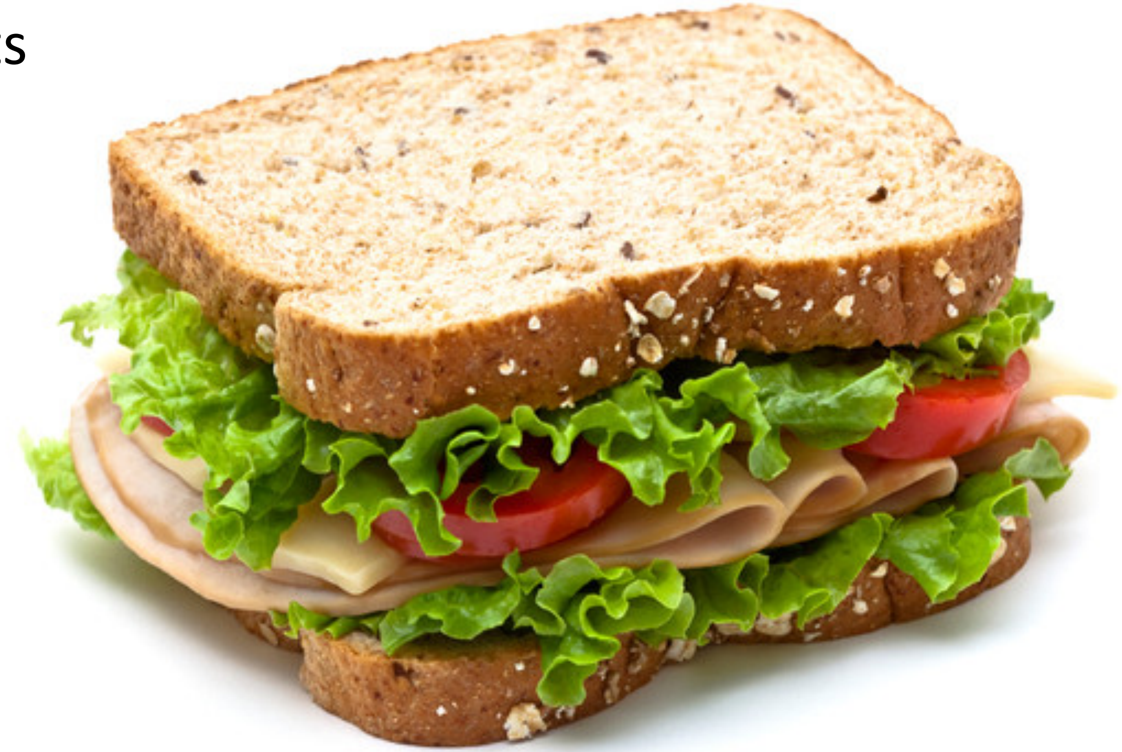
- **How many different sandwiches can Jason make?**



Jason’s sandwich

- Suppose that Jason has the following ingredients to make a sandwich with:
  - White or black bread 2 options
  - Butter, Mayo or Honey Mustard 3 options
  - Romaine Lettuce, Spinach, Kale 3 options
  - Bologna, Ham or Turkey 3 options
  - Tomato or egg slices 2 options
- **How many different sandwiches can Jason make?**
  - $2 \times 3 \times 3 \times 3 \times 2 = 4 \times 27 = 108$



The multiplication rule

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  - Example: $E =$ "$sandwich\ preparation$", $s_1 =$ "$chop\ bread$", $s_2 =$ "$choose\ condiment$", …
- Then, the total number of ways that E can be conducted in is

$$\prod_{i=1}^{k} n_i = n_1 \times n_2 \times \cdots \times n_k$$

A familiar example

- How many subsets are there of a set of 4 elements?
- Example: $\{a, b, c, d\}$
  - a: in or out. 2 choices.
  - b: in or out. 2 choices.
  - c: in or out. 2 choices.
  - d: in or out. 2 choices.

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- Generalization: there are 2^n subsets of a set of size n.
  - But you already knew this.

Permutations

- Consider the string "machinery".
- A **permutation** of "machinery" is **a string which results by re-organizing the characters of "machinery" around**.
  - Examples: choirenima, hcorianemi, machinery (!)
  - Question: How many permutations of "machinery" are there?

Permutations

Permutations

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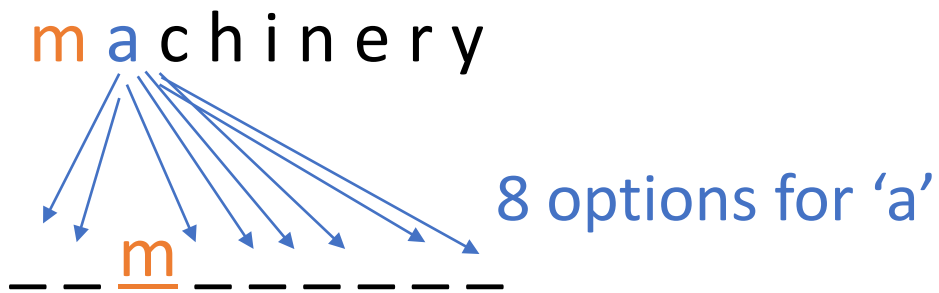
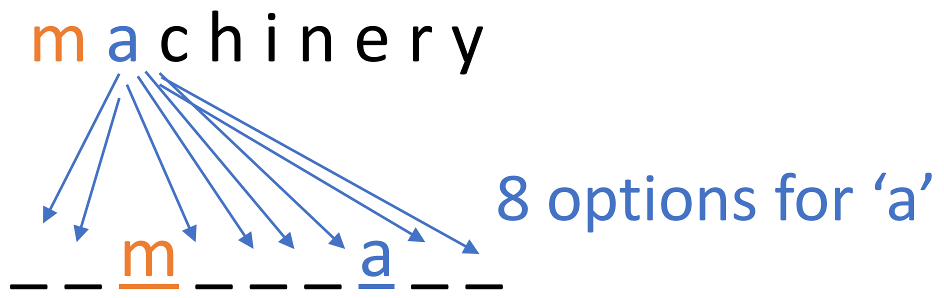


Permutations



Permutations

m a c h i n e r y

7 options for ‘c’…

_ _ m _ _ _ a _ _

Permutations

m a c h i n e r y

7 options for ‘c’…

_ _ m _ _ c a _ _

Permutations

m a c h i n e r y

6 options for 'h'…

_ _ m _ _ c a _ _

Permutations

m a c h i n e r y

h _ m _ _ c a _ _

6 options for ‘h’…

Permutations

m a c h i n e r y

5 options for ‘i’

h _ m _ _ c a _ _

Permutations

m a c h i n e r y

h _ m _ _ c a _ i

5 options for ‘i’

Permutations

m a c h i n e r y

h _ m _ _ c a _ i

4 options for ‘n’

Permutations

m a c h i n e r y

4 options for ‘n’

h _ m _ n c a _ i

Permutations

m a c h i n e r y

h _ m _ n c a _ i

3 options for ‘e’

Permutations

m a c h i n e r y

3 options for ‘e’

h e m _ n c a _ i

Permutations

m a c h i n e r y

h e m _ n c a _ i

2 options for ‘r’

Permutations

m a c h i n e r y

2 options for ‘r’

h e m _ n c a r i

Permutations

m a c h i n e r y

h e m _ n c a r i

1 option for ‘y’

Permutations

m a c h i n e r y

h e m y n c a r i

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Total #possible permutations $= 9 \times 8 \times \cdots \times 2 \times 1 = 9! = 362880$

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That’s a lot! (Original string has length 9)

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In general, for a string of length n we have ourselves $n!$ different permutations!

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- How many permutations are there of this string?
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Permutations

- Now, consider the string "puzzle".
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- Note that two letters in puzzle are the same.
  - Call the first z z_1 and the second z z_2
- So, one permutation of puz_1z_2le is puz_2z_1le
  - But this is clearly equivalent to puz_1z_2le, so we wouldn't want to count it!
  - So clearly the answer is **not** **6! (6 is the length of "puzzle")**
  - **What is the answer?**

Thought Experiment

- Pretend the two ‘z’s in “puzzle” are different, e.g z_1, z_2
  - Then, 6! permutations, as discussed
  - Now we have the “equivalent” permutations, for instance

$$z_1 pul z_2 e$$
$$z_2 pul z_1 e$$

  - We want to **not doublecount** these!

Thought Experiment

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- Then, we need to stop pretending that the ‘z’s are different
  - **Bad news: 6! is overcount ☹**
  - **Good news: 6! is an overcount in a precise way! ☺ Everything is counted exactly twice!**

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  - Answer: $\frac{6!}{2}$

Permutations

- Now, consider the string “scissor”.
- **How many permutations of “scissor” are there?**
- **Note that three** letters in “scissor” are the same.
  - As previously discussed, the answer cannot be **7! (7 is the length of “scissor”)**

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  - Observe all the possible positions of the various 's's:
    - $s_1cis_2s_3or$
    - $s_1cis_3s_2or$
    - $s_2cis_1s_3or$
    - $s_2cis_3s_1or$
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3! = 6 different ways to arrange those 3 's''s

Final answer

- Think of it like this: *How many times can I fit essentially the same string into the number of permutations of the original string?*
- Therefore, the total #permutations when not assume different 's's is

$$\frac{7!}{3!} = \frac{1 \times \cancel{2 \times 3 \times} 4 \times 5 \times 6 \times 7}{1 \cancel{\times 2 \times 3}} = 20 \times 42 = 840$$

Complex overcounting

- Consider now the string "onomatopoeia".
- 12 letters, with 4 'o's, 2 'a's
- Considering the characters being different, we have:

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How many such positionings of the 'o's are possible?

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12

16

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…

$4! = 24$ different ways.

Complex overcounting

- However, we also have the two 'a's to consider!
- Fortunately, those equivalent permutations are simpler to count:

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- Key: **for every one** of these two (equivalent) permutations, we have 4! equivalent permutations because of the 'o's! ***(MULTIPLICATION RULE)***

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- Key: **for every one** of these two (equivalent) permutations, we have 4! equivalent permutations because of the ‘o’s! ***(MULTIPLICATION RULE)***
- Final answer:

$$\#permutations = \frac{12!}{4! \cdot 2!} = \frac{5 \cdot 6 \cdot \ldots \cdot 11 \cdot 12}{2} = 5 \cdot 6^2 \cdot \ldots \cdot 10 \cdot 11 = 9{,}979{,}200$$

Important “pedagogical” note

- In the previous problem, we came up with the quantity

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- **How you should answer in an exam:** $\frac{12!}{4! \cdot 2!}$
- **Don't perform computations, like** $9{,}979{,}200$
  - Helps **you** save time and **us when grading** ☺

For you!

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$$\frac{10!}{2! \cdot 2! \cdot 3!}$$

Don't forget the third 'e'!