Intro to Combinatorics
(“that n choose 2 stuff”)

CMSC 250

More practice

- What about the #non-equivalent permutations for the word

combinatorics

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combinatorics

$$\frac{13!}{2! \cdot 2! \cdot 2!} = \cdots$$

General template

- Total # permutations of a string σ of letters of length *n* where there are n_a $'a's, n_b$ $'b's, n_c$ $'c's, \ldots n_z$ $'z's$

$$\frac{n!}{n_a! \times n_b! \times \cdots \times n_z!}$$

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- Claim: This formula is problematic when some letter (a, b, …, z) is **not** contained in σ

Yes

No

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Remember:

0! = 1

Yes

No

r-permutations

- Warning: permutations (as we've talked about them) are best presented with strings.
- r-permutations: Those are best presented with sets.
  - Note that $r \in \mathbb{N}$
  - So we can have 2-permutations, 3-permutations, etc

r-permutations: Example

- I have ten people.

- My goal: pick three people for a picture, where **order of the people matters.**

r-permutations: Example

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- Examples: shortest-to-tallest or tallest-to-shortest or something-in-between

r-permutations: Example

- I have ten people.

- My goal: pick three people for a picture, where **order of the people matters.**
- Examples: Jenny-Fred-Bob or Fred-Jenny-Bob or Fred-Bob-Jenny

r-permutations: Example

- I have ten people.

- My goal: pick three people for a picture, where **order of the people matters.**
- In how many ways can I pick these people?

r-permutations: Example

r-permutations: Example

I need three people for this photo. You guys figure out your order.

r-permutations: Example

10 ways to pick the first person…

. . .

I need three people for this photo. You guys figure out your order.

___ ___ ___

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10 ways
to pick
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r-permutations: Example

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9 ways to pick the **second** person…

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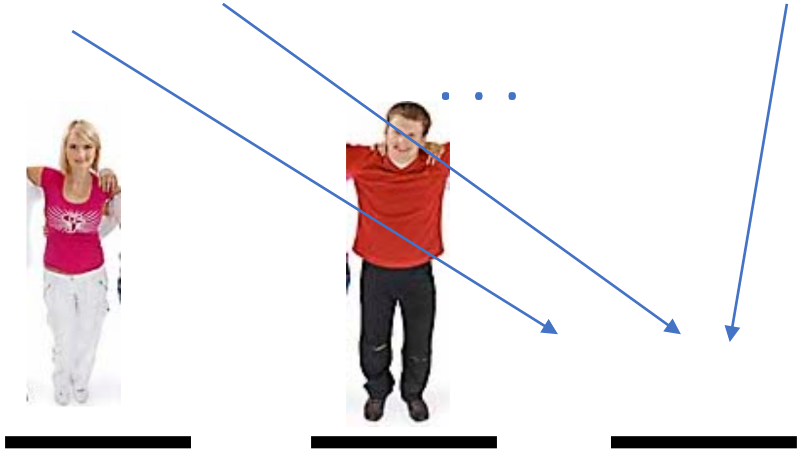

8 ways to pick the **third** person…

r-permutations: Example

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For a total of $10 \times 9 \times 8 = 720$ ways.

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Note: $10 \times 9 \times 8 = \frac{10!}{(10-3)!}$

Example on Books

- Clyde has the following books on his bookshelf
  - Epp, Rosen, Hughes, Bogart, Davis, Shaffer, Sellers, Scott
- Jason wants to borrow any 5 of them and read them in the order he picks them in.
- In how many ways can Jason get smart by reading those books?

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$$\frac{8!}{(8-5)!} = \frac{8!}{3!}$$

General formula

- Let $n, r \in \mathbb{N}$ such that $0 \leq r \leq n$. The total ways in which we can select r elements from a set of n elements where order matters is equal to:

$$P(n,r) = \frac{n!}{(n-r)!}$$

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"P" for permutation. This quantity is known as the r-permutations of a set with n elements.

Pop quizzes

1) $P(n, 1) = \cdots$ | 0 | 1 | n | $n!$

Pop quizzes

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- Two ways to convince yourselves:
  - Formula: $\frac{n!}{(n-1)!} = n$
  - Semantics of r-permutations: In how many ways can I pick 1 element from a set of n elements? Clearly, I can pick any one of n elements, so n ways.

Pop quizzes

2) $P(n, n) = \cdots$

0	1	n	$n!$

Pop quizzes

2) $P(n, n) = \cdots$ | 0 | 1 | n | $n!$

- Again, two ways to convince ourselves:
  - Formula: $\frac{n!}{(n-n)!} = \frac{n!}{0!}$
  - Semantics: $n!$ ways to pick all of the elements of a set and put them in order!

Pop quizzes

3) $P(n,0) = \cdots$

0	1	n	$n!$

Pop quizzes

3) $P(n, 0) = \cdots$

0	1	n	$n!$

- Again, two ways to convince ourselves:
  - Formula: $\frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$
  - Semantics: Only one way to pick nothing: just pick nothing and leave!

Practice

1. How many MD license plates are possible to create?

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Remember these phrases!

Combinations (that "n choose r" stuff)

- Earlier, we discussed this example:

I need three people for this photo. You guys figure out your order.

- Our goal was to pick three people for a picture, where **order of the people mattered.**

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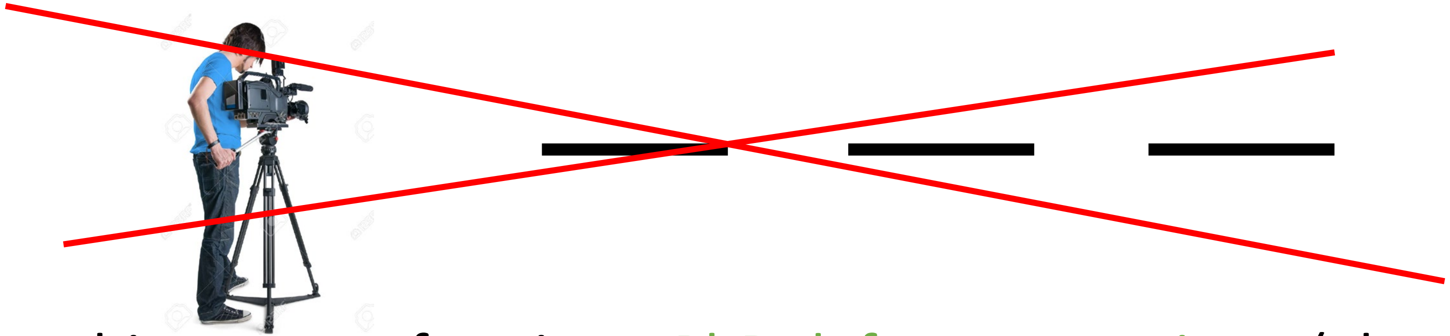

- We now change this setup to forming a PhD defense committee (also 3 people).
- In this setup, does order matter?

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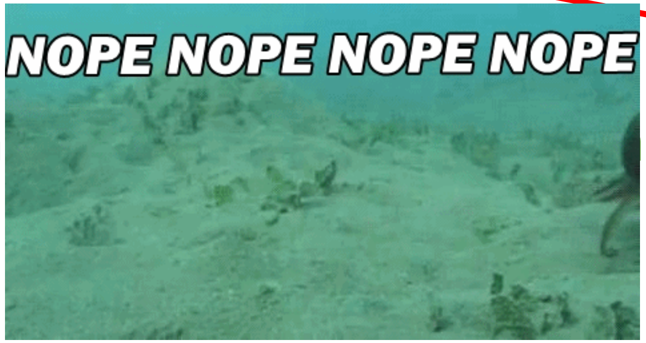

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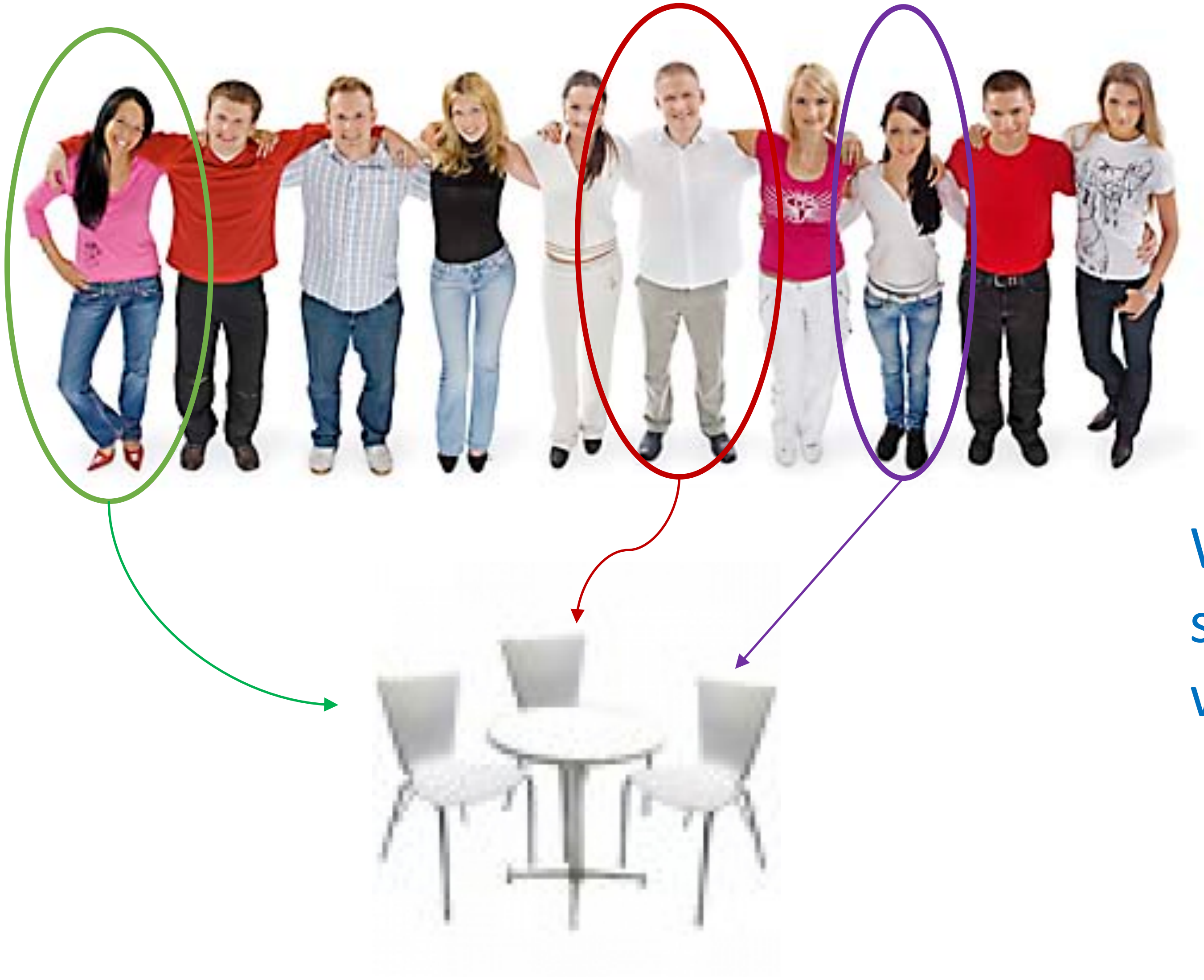

We can make this selection in $P(10, 3)$ ways…

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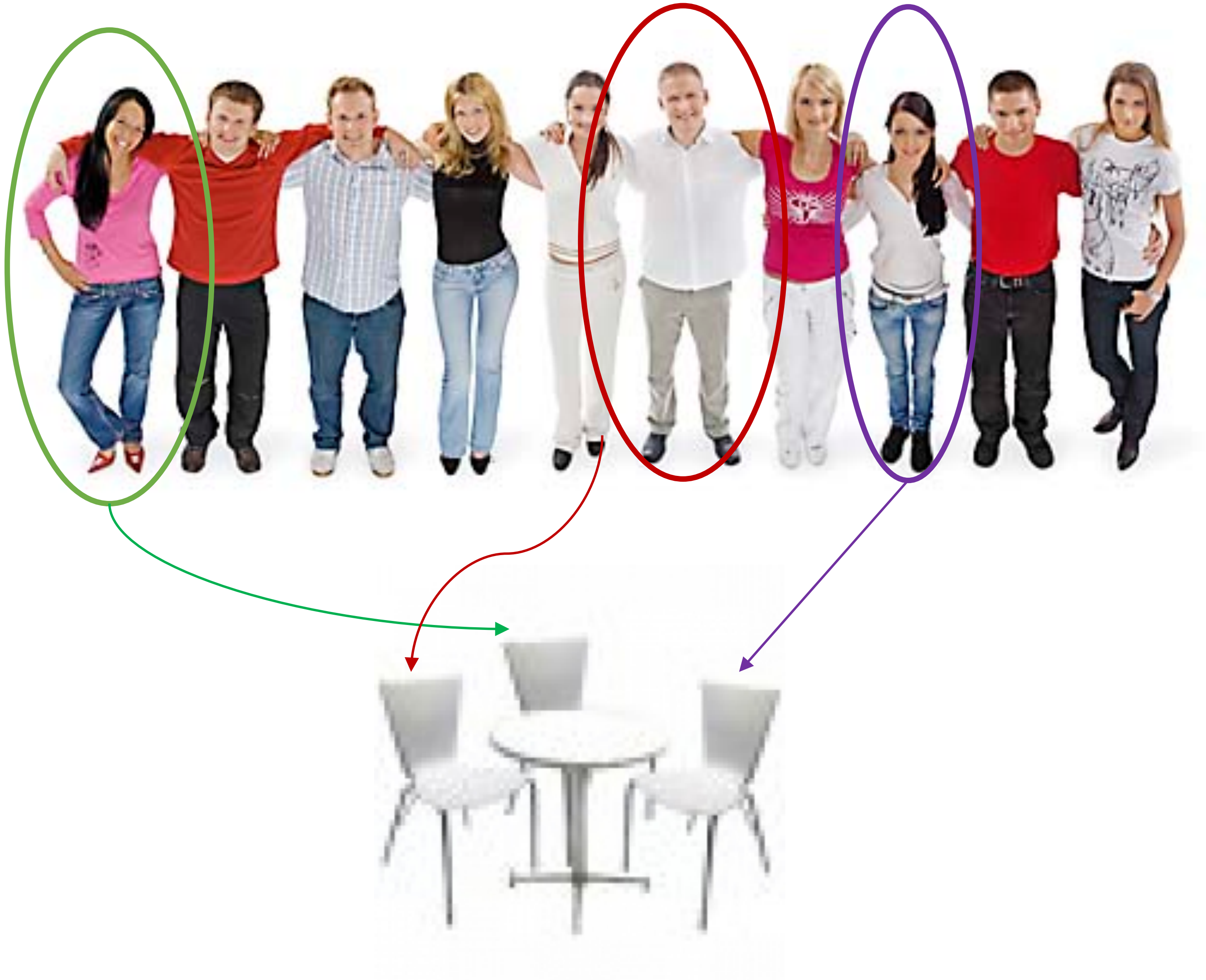

We can make this selection in $P(10,3)$ ways… but **since order doesn't matter**, we have 3! permutations of these people that are equivalent.

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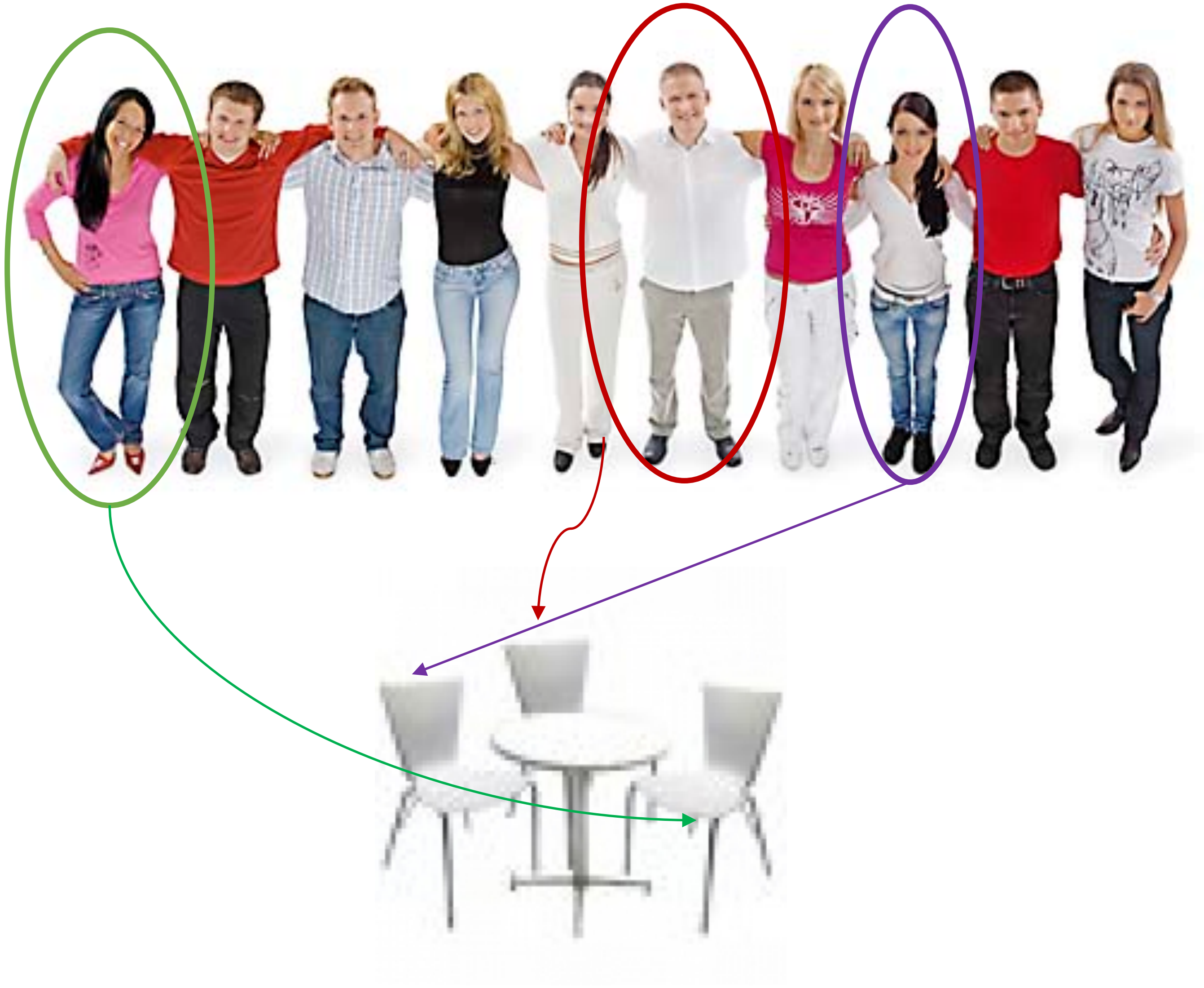

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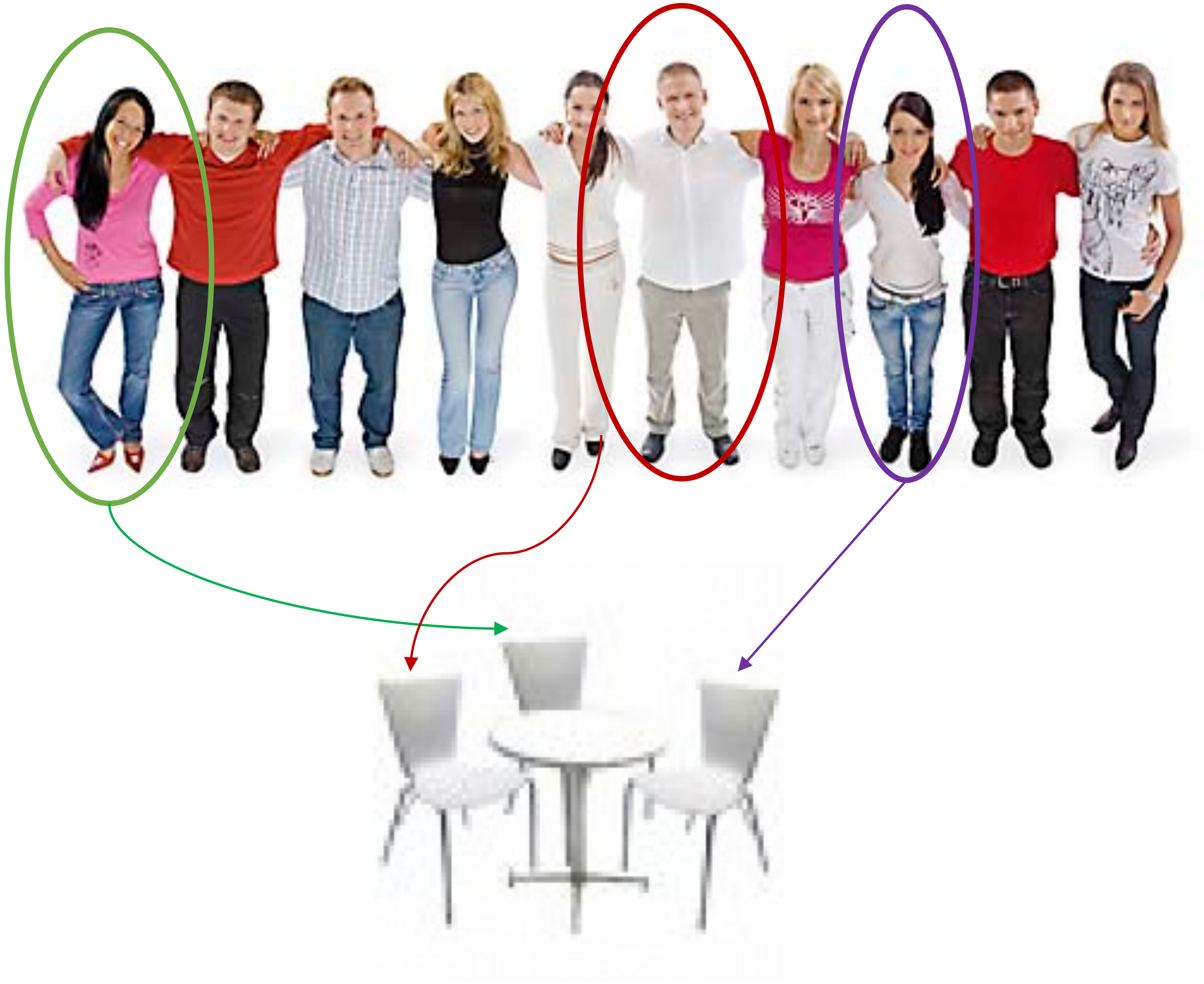

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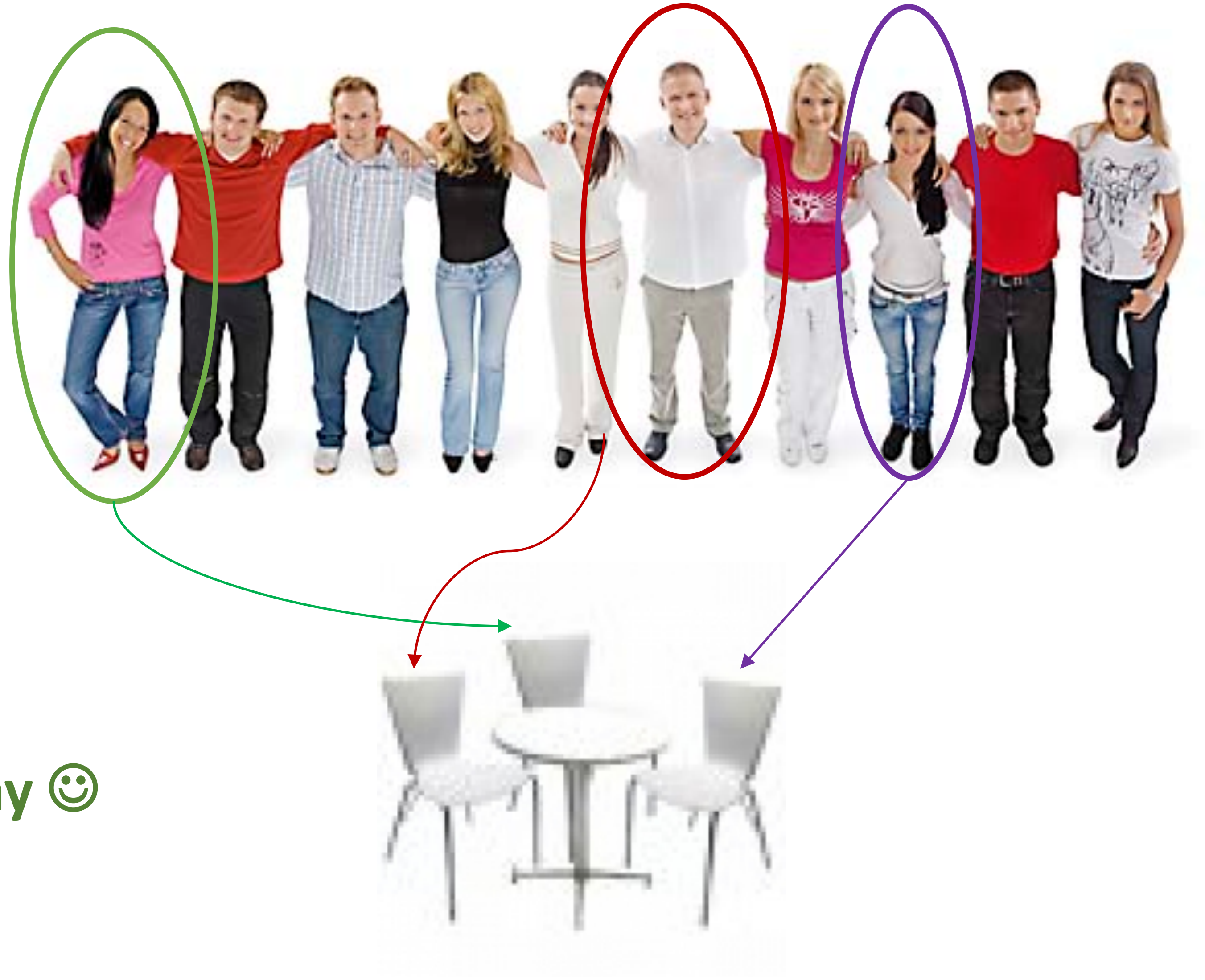

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In a precise way ☺

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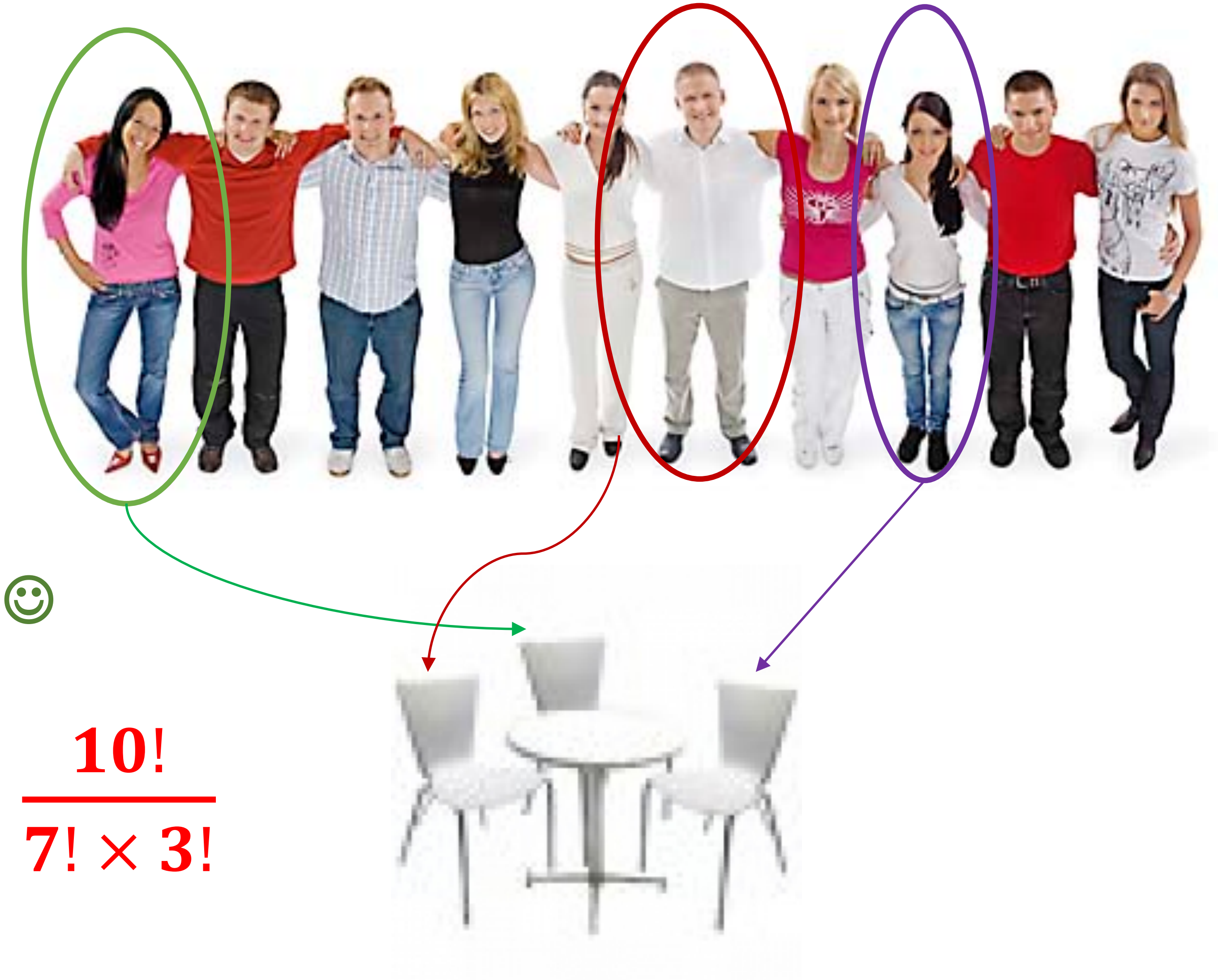

We can make this selection in $P(10,3)$ ways… but **since order doesn’t matter**, we have 3! permutations of these people that are equivalent.

Overcount ☹

In a precise way ☺

$$\frac{P(10,3)}{3!} = \frac{10!}{7! \times 3!}$$

Closer analysis of example

- Note that essentially we are asking you: Out of a set of 10 people, how many subsets of 3 people can I retrieve?

$\binom{n}{r}$ notation

- The quantity

$$\frac{P(10,3)}{3!}$$

is the number of *3-combinations* from a set of size 10, denoted thus:

$$\binom{n}{3}$$

and pronounced “n choose 3”.

$\binom{n}{r}$ notation

- Let $n, r \in \mathbb{N}$ with $0 \leq r \leq n$
- Given a set A of size n, the total number of subsets of A of size r is:

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

$\binom{n}{r}$ notation

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- Given a set *A* of size n, the total number of subsets of *A* of size r is:

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- Pop quiz: $(\forall n, r \in \mathbb{N})[(0 \leq r \leq n) \Rightarrow (\binom{n}{r} \leq P(n,r))]$

True

False

$\binom{n}{r}$ notation

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- Pop quiz: $(\forall n, r \in \mathbb{N})[(0 \leq r \leq n) \Rightarrow (\binom{n}{r} \leq P(n,r))]$

Recall that

$\binom{n}{r} = \frac{P(n,r)}{r!}$ and $r! \geq 1$

True

False

Quiz

Quiz

1	n	$n!$	Sth else

1. $\binom{n}{1} =$

Quiz

1	n	$n!$	Sth else

1. $\binom{n}{1} = n$

Quiz

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1. $\binom{n}{1} = n$
2. $\binom{n}{n} =$

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1	n	$n!$	Sth else

1. $\binom{n}{1} = n$
2. $\binom{n}{n} = 1$ (Note how this differs from $P(n,n) = n!$)

Quiz

1	n	$n!$	Sth else

1. $\binom{n}{1} = n$
2. $\binom{n}{n} = 1$ (Note how this differs from $P(n,n) = n!$)
3. $\binom{n}{0} =$

Quiz

1 | n | $n!$ | Sth else

1. $\binom{n}{1} = n$
2. $\binom{n}{n} = 1$ (Note how this differs from $P(n,n) = n!$)
3. $\binom{n}{0} = 1$