Discrete Probability

CMSC 250

Informal definition of probability

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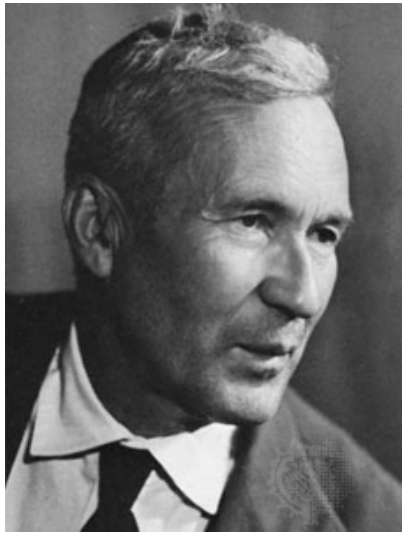
- This definition is owed to Andrey Kolmogorov, and assumes *that all possibilities are equally likely!*



First examples

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$\frac{1}{3}$

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  - What is the probability that I don't get any heads?
  - Why?
    - Set of different *events?*
      - $\{HHH, HHT, HTH, HTT, THH, THT, TTH, \boldsymbol{TTT}\}$ (8 of them)
    - Set of events with **no heads:**
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$\frac{1}{9}$	Something else

Implicit assumption: all individual outcomes (HHH, HHT, HTH, ….) ***are considered equally likely (probability 1/8)***

Practice

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  - Probability that I hit seven = ?

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      - $\{(1,1),(1,2),\ldots,(6,1)\}$ (36 of them)
    - Set of events where we hit 7.
      - $\{(2,5),(5,2),(3,4),\ (4,3),(1,6),\ (6,1)\}$ (6 of them)
    - Hence the answer: $\frac{6}{36} = \frac{1}{6}$

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  - Probability that I hit two= ?
    - Same procedure

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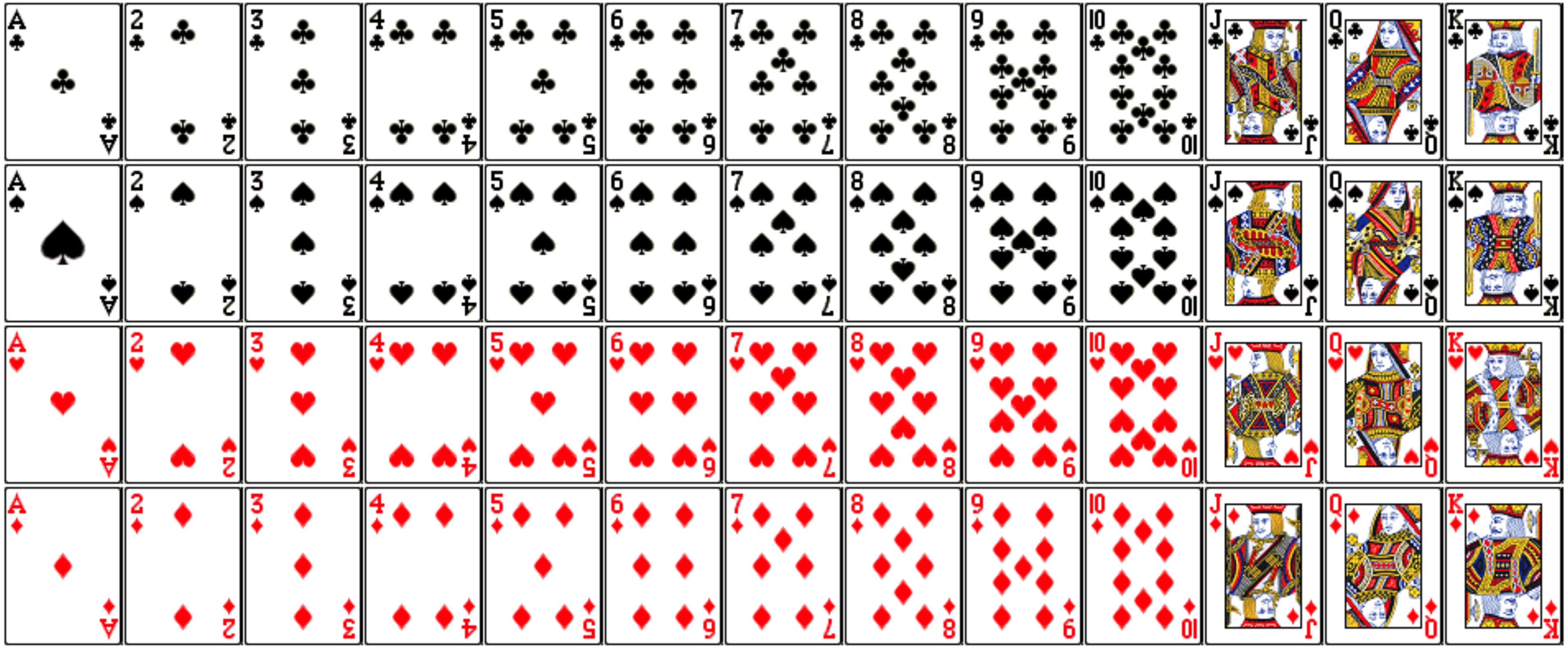
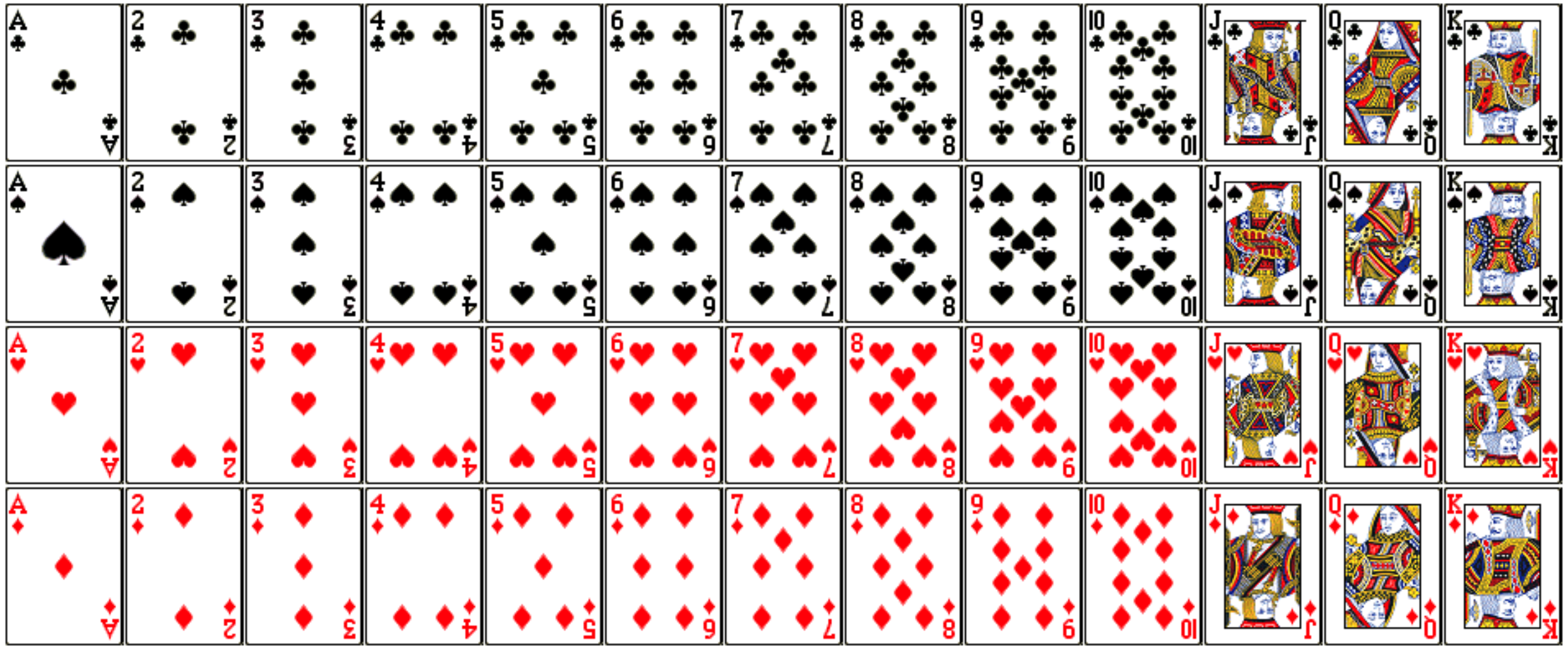
Poker Practice

- Full deck = 52 cards, 13 of each suit:



Poker Practice

- Full deck = 52 cards, 13 of each suit:
- **Flush**: 5 cards of the same suit
- What is the probability of getting a flush?



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- So, probability of being dealt a flush is

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- How likely is this?
  - Not at all likely: $\approx 0.002 = 0.2\%$ ☹

Likelihood of a straight

- Straights are 5 cards of ***consecutive rank***
  - Ace can be ***either end*** (high or low)
  - ***No wrap-arounds*** (e.g Q K A 2 3, suits don't matter)
- What is the probability that we are dealt a straight?

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That's $10 * 4^5$ ways. So, probability of a straight = $\frac{10*4^5}{\binom{52}{5}}$

Caveat on flushes

- Wikipedia says we're wrong about flushes!
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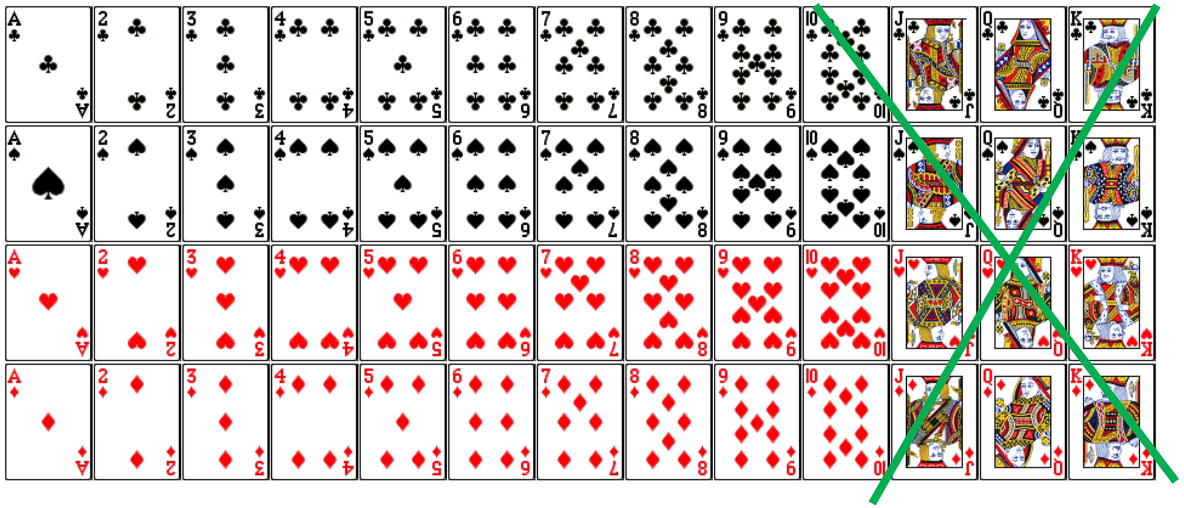
  - Hands like these are called straight flushes and Wikipedia does not include them.
  - ***How many straight flushes are there?***
  - 40. Here's why:
    - Pick rank: A through 10 (higher ranks don't allow straights) in 10 ways
    - Pick suit in 4 ways



Probability of non-straight flush…

$$\frac{4 * \binom{13}{5} - 40}{\binom{52}{5}} = 0.001965$$

- This is how Wikipedia defines the probability of a flush. ☺

Probability of a straight flush...

$$\frac{40}{\binom{52}{5}} = 0.0000138517$$

Probability of a straight flush...

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The expected # hands you need to play to get a straight flush is then

$$\left\lceil \frac{1}{0.0000138517} \right\rceil = 72{,}194$$

Same caveat for straights

- From the #straights we computed we will have to subtract the 40 possible straight flushes to get…

$$\frac{10 * 4^5 - 40}{\binom{52}{5}} = 0.003925$$

Same caveat

- From the #straights we computed we will have to subtract the 40 possible straight flushes to get…

$$\frac{10*4^5-40}{\binom{52}{5}} = 0.003925 > 0.001965 = \text{probability of flush}$$

- *Flushes, being more rare, beat straights in poker.*

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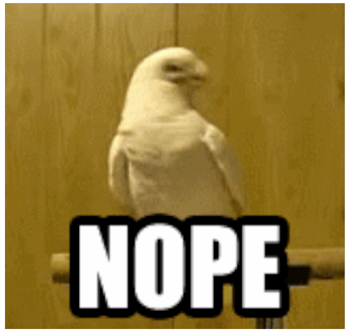

Is this accurate?

Severe overcount!

Yes

No

Don’t count better hands!

- In the computation before, we included:
  - 3-of-a-kind
  - 4-of-a-kind
  - etc
- To properly compute, we would have to subtract **all** better hands possible with at least one pair.