Discrete Probability

CMSC 250

Joint probability (“AND” of two events)

- The probability that two events A and B occur **simultaneously** is known as the **joint** probability of A and B and is denoted in a number of ways:
    - $P(A \cap B)$ (Most useful from a set-theoretic perspective; we’ll be using this)
    - $P(A, B)$ (One sees this a lot in Physics books)
    - $P(AB)$ (Perhaps most convenient, therefore most common)

Calculating joints

- Probability that the first coin toss is heads and the second coin toss is tails

Calculating joints

- Probability that the first coin toss is heads and the second coin toss is tails $\frac{1}{2} \times \frac{1}{2}$

Calculating joints

- Probability that the first coin toss is heads and the second coin toss is tails $\frac{1}{2} \times \frac{1}{2}$
- Probability that the first die is at most a 2 and the second one is 5 or 6

Calculating joints

- Probability that the first coin toss is heads and the second coin toss is tails $\frac{1}{2} \times \frac{1}{2}$
- Probability that the first die is at most a 2 and the second one is 5 or 6
  - # outcomes of die roll is 6
  - # outcomes where first die is at most 2 is 2
    - Hence, probability of first die roll being at most 2 is $\frac{1}{3}$

Calculating joints

- Probability that the first coin toss is heads and the second coin toss is tails $\frac{1}{2} \times \frac{1}{2}$
- Probability that the first die is at most a 2 and the second one is 5 or 6
  - # outcomes of die roll is 6
  - # outcomes where first die is at most 2 is 2
    - Hence, probability of first die roll being at most 2 is $\frac{1}{3}$
  - Similarly, probability of second die roll being 5 or 6 is $\frac{1}{3}$.
  - Hence, probability that **both** events happen (joint probability) is $\frac{1}{9}$.

Calculating joints

- Jason's going to flip a coin and then pick a card from a 52-card deck.
  - Probability that the coin is heads and the card has rank 8?

$\frac{1}{2}$	$\frac{1}{26}$	$\frac{1}{32}$	Something else

Calculating joints

- Jason's going to flip a coin and then pick a card from a 52-card deck
  - Probability that the coin is heads and the card has rank 8?

$\frac{1}{2}$	$\frac{1}{26}$	$\frac{1}{32}$	Something else

- This is because $P(coin = H) = \frac{1}{2}$ and $P(card\_rank = 8) = \frac{4}{52} = \frac{1}{13}$
  - So their joint probability is $\frac{1}{2} \times \frac{1}{13} = \frac{1}{26}$

The law of joint probability

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = \prod_{i=1}^{n} P(A_i)$$

The law of joint probability

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = \prod_{i=1}^{n} P(A_i)$$

- Unfortunately, this "law" is not always applicable!
- It is applicable only when all the different events A_i are *independent* (sometimes called *marginally independent*) of each other.
- Let's look at an example.

Calculating joints of dependent events

- Probability that a die is even and that it is 2.

Calculating joints of dependent events

- Probability that a die is even and that it is 2.
  - Probability that the die is even = $\frac{1}{2}$

Calculating joints of dependent events

- Probability that a die is even and that it is 2.
  - Probability that the die is even = $\frac{1}{2}$
  - Probability that the die is two = $\frac{1}{6}$

Calculating joints of dependent events

- Probability that a die is even and that it is 2.
  - Probability that the die is even = $\frac{1}{2}$
  - Probability that the die is two = $\frac{1}{6}$
  - Probability the die is even **and** the die is two = $\frac{1}{12}$???

Calculating joints of dependent events

- Probability that a die is even and that it is 2.
  - Probability that the die is even = $\frac{1}{2}$
  - Probability that the die is two = $\frac{1}{6}$
  - Probability the die is even **and** the die is two = $\frac{1}{12}$???
  - ***NO!***
    - What is the probability that the die is even and the die is 2?

$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$

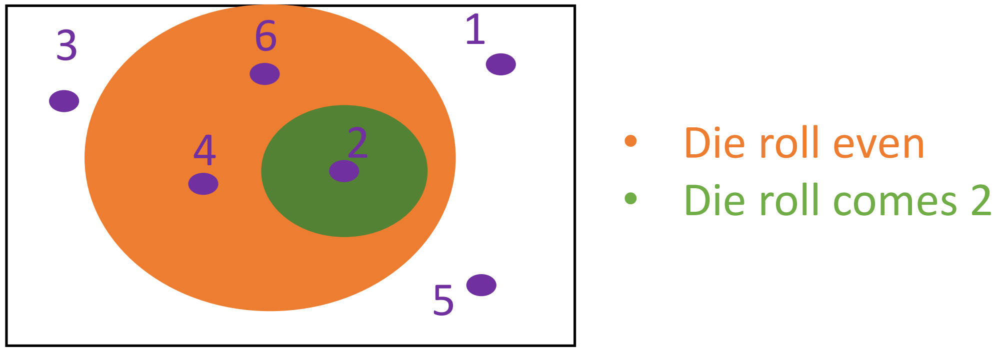
Calculating joints of dependent events

- Probability that a die is even and that it is 2.
  - Probability that the die is even = $\frac{1}{2}$
  - Probability that the die is two = $\frac{1}{6}$
  - Probability the die is even **and** the die is two = $\frac{1}{12}$???
  - ***NO!***
    - What is the probability that the die is even and the die is 2?

$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$

Set-theoretic interpretation

- Notice that the event A: “Die roll is even” is a superset of the event B: “Die roll comes 2”



- Since $A \cap B = A$, $P(A \cap B) = P(A) = \frac{1}{6}$

Calculating joints

- The University of Southern North Dakota offers a Discrete Mathematics Course where the possible grades are A through G. (No + or -)
- What is the probability that Jason gets **both** an A and a G in that course?

Calculating joints

- The University of Southern North Dakota offers a Discrete Mathematics Course where the possible grades are A through G. (No + or -)
- What is the probability that Jason gets **both** an A and a G in that course?
  - Clearly, it **can't** be

(probability Jason gets an A) X (probability Jason gets a B) = $\frac{1}{7} \times \frac{1}{7} = \frac{1}{49}$

Calculating joints

- The University of Southern North Dakota offers a Discrete Mathematics Course where the possible grades are A through G. (No + or -)
- What is the probability that Jason gets **both** an A and a G in that course?
  - Clearly, it **can't** be

~~*(probability Jason gets an A) X (probability Jason gets a B) =* $\frac{1}{7} \times \frac{1}{7} = \frac{1}{49}$~~

Calculating joints

- The University of Southern North Dakota offers a Discrete Mathematics Course where the possible grades are A through G. (No + or -)
- What is the probability that Jason gets **both** an A and a G in that course?
  - Clearly, it **can't** be

~~*(probability Jason gets an A) X (probability Jason gets a B) =* $\frac{1}{7} \times \frac{1}{7} = \frac{1}{49}$~~

  - It is **0**. Those two events cannot happen ***jointly*!**

Calculating joints

- The University of Southern North Dakota offers a Discrete Mathematics Course where the possible grades are A through G. (No + or -)
- What is the probability that Jason gets **both** an A and a G in that course?
  - Clearly, it **can't** be

~~*(probability Jason gets an A) x (probability Jason gets a B)* = $\frac{1}{7} \times \frac{1}{7} = \frac{1}{49}$~~

  - It is **0**. Those two events cannot happen ***jointly*!**
  - Events such as these are called *disjoint* or *mutually disjoint*.

Set-theoretic interpretation

- A = "Jason gets an A in USND's 250"
- G="Jason gets a G in USND's 250"

- Note that $A \cap G = \emptyset$, so there are no common outcomes.
  - So $P(A \cap G) = 0$

Calculating joints

- I have my original die again.
  - Probability that it comes up 1, 2 or 3 = $\frac{1}{2}$
  - Probability that it comes up 3, 4 or 5 = $\frac{1}{2}$
  - What is the probability that it comes up 1, 2 or 3 **and** 3, 4 or 5?

Calculating joints

- I have my original die again.
  - Probability that it comes up 1, 2 or 3 = $\frac{1}{2}$
  - Probability that it comes up 3, 4 or 5 = $\frac{1}{2}$
  - What is the probability that it comes up 1, 2 or 3 **and** 3, 4 or 5?

$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$

Calculating joints

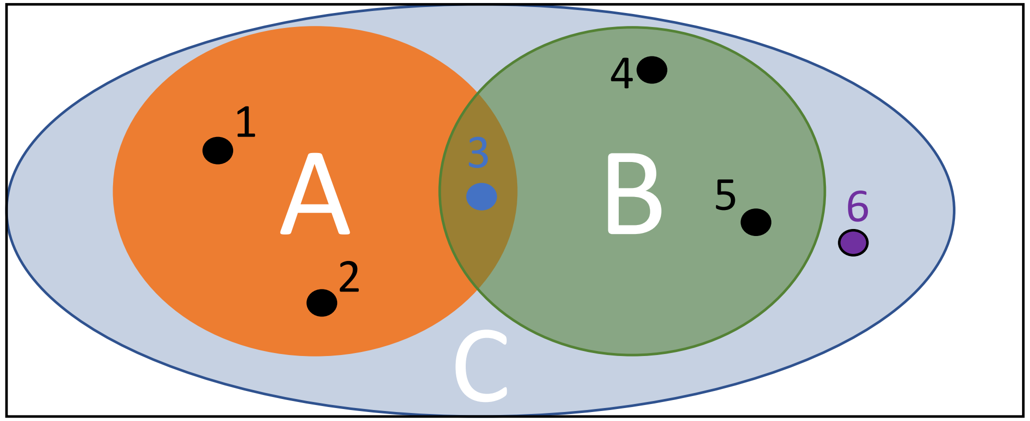
- I have my original die again.
  - Probability that it comes up 1, 2 or 3 = $\frac{1}{2}$
  - Probability that it comes up 3, 4 or 5 = $\frac{1}{2}$
  - What is the probability that it comes up 1, 2 or 3 **and** 3, 4 or 5?

$\frac{1}{6}$ $\frac{1}{5}$ $\frac{1}{4}$ $\frac{1}{3}$

  - Note that the only common outcome between the two events is **3**, which can come up only **once** out of **six** possibilities.

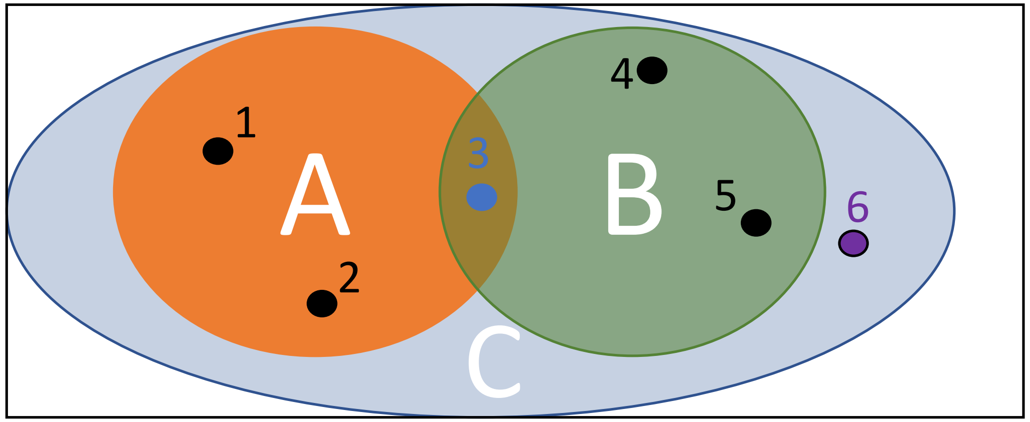
Set-theoretic interpretation

- Let A = dice comes up 1, 2, or 3
- Let B = dice comes up 3, 4, or 5
- Let C = dice comes up 1, 2, 3, 4, 5 OR 6



Set-theoretic interpretation

- Let A = dice comes up 1, 2, or 3
- Let B = dice comes up 3, 4, or 5
- Let C = dice comes up 1, 2, 3, 4, 5 OR 6



- Then, probability that the dice comes up 3 = $\frac{1}{6}$

Independent events *(informally)*

- Two events are independent if **one does not influence the other.**
- Examples:
  - The event E1 = “first coin toss” and E2 = “second coin toss”
  - With the same die, the events E1 = “roll 1”, E2 = “roll 2”, E3 = “roll 3”
  - Jason flips a coin and then picks a card.
- Counter-examples:
  - E1 = “Die is even”, E2=“Die is 6”
  - E1= “Grade in 250” and “Passing 250”

Law of joint probability *(informally)*

- Two events are independent if **one does not influence the other.**
  - This definition is a but too informal, so mathematicians tend to avoid it.
- Formally, we define that A and B are independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

Disjoint or independent?

1. E_1 = "It rains in College Park, MD today"
   E_2 = "It rains in Athens, Greece today"

Disjoint | Independent | Both | Neither

Disjoint or independent?

1. $E_1 =$ "It rains in College Park, MD today"
   $E_2 =$ "It rains in Athens, Greece today"

Disjoint | Independent | Both | Neither

Weather is weird!

Disjoint or independent?

1. E_1 = "It rains in College Park, MD today"
   E_2 = "It rains in Athens, Greece today"

   Disjoint | Independent | Both | Neither

   Weather is weird!

2. E_1 = "It rains in College Park, MD today"
   E_2 = "It is sunny in College Park, MD today"

   Disjoint | Independent | Both | Neither

Disjoint or independent?

1. E_1 = "It rains in College Park, MD today"
   E_2 = "It rains in Athens, Greece today"

   Disjoint | Independent | Both | Neither

   Weather is weird!

2. E_1 = "It rains in College Park, MD today"
   E_2 = "It is sunny in College Park, MD today"

   Disjoint | Independent | Both | Neither

Disjoint or independent?

1. E_1 = "It rains in College Park, MD today"
   E_2 = "It rains in Athens, Greece today"

   Disjoint | Independent | Both | Neither

   Weather is weird!

2. E_1 = "It rains in College Park, MD today"
   E_2 = "It is sunny in College Park, MD today"

   Disjoint | Independent | Both | Neither

3. E_1 = Die **#1** comes at most 4
   E_2 = Die **#2** comes at least 5

   Disjoint | Independent | Both | Neither

Disjoint or independent?

1. E_1 = "It rains in College Park, MD today"
   E_2 = "It rains in Athens, Greece today"

   Disjoint | Independent | Both | Neither

   Weather is weird!

2. E_1 = "It rains in College Park, MD today"
   E_2 = "It is sunny in College Park, MD today"

   Disjoint | Independent | Both | Neither

3. E_1 = Die **#1** comes at most 4
   E_2= Die **#2** comes at least 5

   Disjoint | Independent | Both | Neither

Disjoint or independent?

1. E_1 = "It rains in College Park, MD today"
   E_2 = "It rains in Athens, Greece today"

   Disjoint | Independent | Both | Neither

   Weather is weird!

2. E_1 = "It rains in College Park, MD today"
   E_2 = "It is sunny in College Park, MD today"

   Disjoint | Independent | Both | Neither

3. E_1 = Die **#1** comes at most 4
   E_2= Die **#2** comes at least 5

   Disjoint | Independent | Both | Neither

4. E_1 = Student gets a C
   E_2 = Student passes the class

   Disjoint | Independent | Both | Neither

Disjoint or independent?

1. E_1 = "It rains in College Park, MD today"
   E_2 = "It rains in Athens, Greece today"

   Disjoint | Independent | Both | Neither

   Weather is weird!

2. E_1 = "It rains in College Park, MD today"
   E_2 = "It is sunny in College Park, MD today"

   Disjoint | Independent | Both | Neither

3. E_1 = Die #1 comes at most 4
   E_2 = Die **#2** comes at least 5

   Disjoint | Independent | Both | Neither

4. E_1 = Student gets a C
   E_2 = Student passes the class

   Disjoint | Independent | Both | Neither

Disjoint Probability (“OR” of two events)

- Jason rolls two dice.
  - **What is the probability that he rolls a 7 or a 9?**

Disjoint Probability ("OR" of two events)

- Jason rolls two dice.
  - **What is the probability that he rolls a 7 or a 9?**
  - #Ways to roll a 7 is 6.
  - #Ways to roll a 9 is 4: (6, 3), (5, 4), (4, 5), (3, 6)
  - #Ways to roll a 7 OR a 9 is then 10.
  - Therefore, the probability is $\frac{10}{36} = \frac{5}{18}$
  - Key: Rolling a 7 and a 9 are **disjoint events**.

Disjoint probability (“OR”)

- 52-card deck
- Probability of drawing a face card (J, Q, K) or a heart

Disjoint probability (“OR”)

- 52-card deck
- Probability of drawing a face card (J, Q, K) or a heart
  - **Are these disjoint?**

Disjoint probability ("OR")

- 52-card deck
- Probability of drawing a face card (J, Q, K) or a heart
  - **Are these disjoint?**
  - ***NO,*** for example, **Queen of hearts**
- How big is $Face\_Card \cup Hearts$?

Disjoint probability (“OR”)

- 52-card deck
- Probability of drawing a face card (J, Q, K) or a heart
  - **Are these disjoint?**
  - ***NO,*** for example, **Queen of hearts**
- How big is $Face\_Card \cup Hearts$ (abbrv. F, H below)?
  - Use law of inclusion / exclusion!

$$|F \cup H| = |F| + |H| - |F \cap H| = 12 + 13 - \mathbf{3} = \mathbf{22}$$

Disjoint probability ("OR")

- 52-card deck
- Probability of drawing a face card (J, Q, K) or a heart
  - **Are these disjoint?**
  - ***NO,*** for example, **Queen of hearts**
- How big is $Face\_Card \cup Hearts$ (abbrv. F, H below)?
  - Use law of inclusion / exclusion!

$$|F \cup H| = |F| + |H| - |F \cap H| = 12 + 13 - \mathbf{3} = \mathbf{22}$$

- So probability = $\frac{22}{52} = \frac{11}{26}$.

Alternative viewpoint

- $P(F) = \frac{12}{52}$
- $P(H) = \frac{13}{52}$
- $P(F \cap H) = \frac{3}{52}$
- $P(F \cup H) = P(F) + P(H) - P(F \cap H)$

Probability of unions

$$P(A \cup B) = P(A) + P(B) - P\,(A \cap B)$$

- If A and B are independent, we have

$$P(A \cup B) = P(A) + P(B) - P\,(A) \cdot P(B)$$

- If A and B are disjoint, we have

$$P(A \cup B) = P(A) + P(B)$$

Probability of unions of 3 sets

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

- If A, B and C are pairwise independent , we have :

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A) \cdot P(B) - P(B) \cdot P(C) - P(A) \cdot P(C) + P(A \cdot B \cdot C)$$

- If A, B and C are pairwise disjoint (so $A \cap B = A \cap C = B \cap C = \emptyset$, so clearly $A \cap B \cap C = \emptyset$), we have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

Recap: “Disjoint” vs “independent”

- Friends don’t let friends get confused between “disjoint” and “independent”!

Disjoint	Independent
Has a set-theoretic interpretation!	Has a causality interpretation!
Means that $P(A \cap B) = 0$	Means that $P(A \cap B) = P(A) \cdot P(B)$
Means that $P(A \cup B) = P(A) + P(B)$	Means that $P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$

Multiplication Rule for Independent Events

- If you flip a coin 5 times, what is the probability that it will be heads every time?

- In Monopoly, you go to jail if you roll “doubles” three times in a row. What is the probability of this happening on a given turn?

- Stephen Curry is the NBA player with the highest career free throw percentage, which is almost exactly 90%. If Stephen went to the line 10 times, what is the probability that he would sink all ten free throws?

Probabilities with Compliements

$$P(E') = 1 - P(E)$$

Probabilities with Compliments

- What is the probability that your 4-digit PIN has at least one repeated digit?
- What is the probability that your Maryland license plate has at least one 7? (Guess first for fun!)
- A certain medication is 95% effective. (That means that if used properly for 1 year it will work 95% of the time.) What is the chance of at least one failure over a 10 year interval?

- A group of 5 students are to be seated in 5 chairs. What is the probability that James ends up sitting next to Nancy? (Guess first!)
- What is the probability that James does NOT end up sitting next to Nancy? (Easy question…)
- If the group consists of 3 men and 2 women, what is the probability that all of the men will end up sitting next to each other? (Guess first!)

Decision Tree

- People= {Alice, Bob, Carolyn, Dan}
- Need to be appointed as president, vice-president, and treasurer, and nobody can hold more than one office
  - how many ways can it be done with no restrictions? (Easy)
  - how many ways can it be done if Alice doesn't want to be president? (Pretty easy)
  - how many ways can it be done if Alice doesn't want to be president, and only Bob and Dan are willing to be vice-president? (Harder)