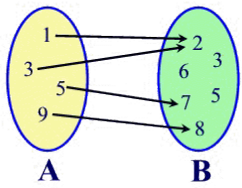
CMSC 250
Discrete Structures

Functions

Function

- A **function** assigns members of one set (the **domain**) to members of another set (**co-domain**)
- The **range** is the subset of the co-domain that gets "hit"

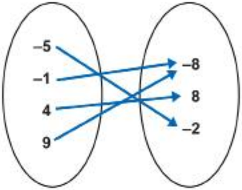
$f : A \rightarrow B$



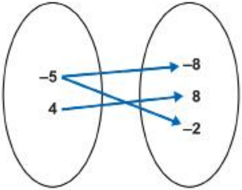
Function

- A member of the domain can only be assigned to *one* member of the co-domain

Are these functions?



Mapping A



Mapping B

Function

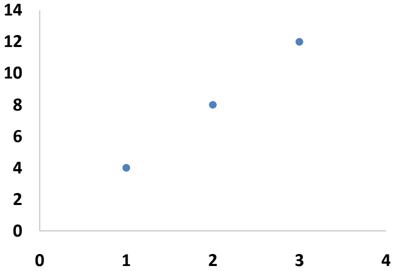
There are many ways to express functions:

Let $A = \{1, 2, 3\}$ $B=\{4, 8, 12\}$

$f : A \rightarrow B$ such that for all $a \in A$, $f(a) = 4a$

$f : A \rightarrow B$ such that $a \mapsto 4a$

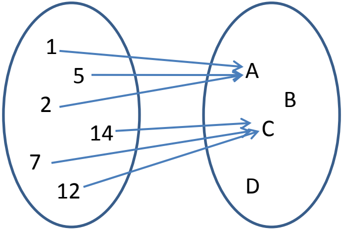
$f = \{<1, 4>, <2, 8>, <3, 12>\}$



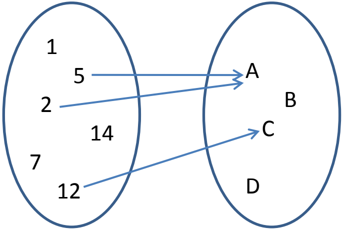
Total vs. Partial

- A **total function** assigns *every* member of the domain to an element of the co-domain
- A **partial function** may not assign every member of the domain

Total Function



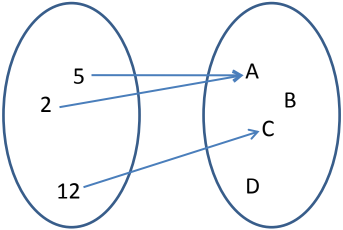
Partial Function



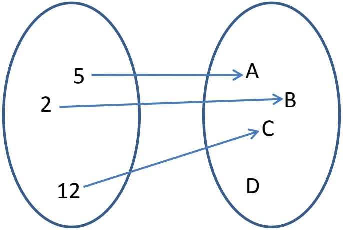
Injective Function

A function is **one-to-one** or **injective** if every element of the range is associated with *exactly one* element from the domain.

Not an Injection



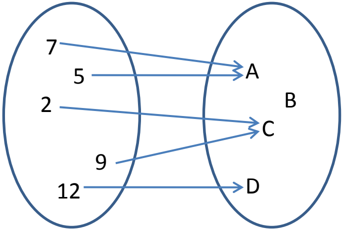
Injection



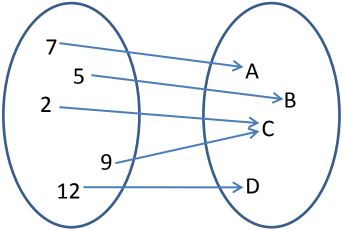
Surjective Function

A function is **onto** or **surjective** if the range is equal to the entire co-domain.

Not a Surjection



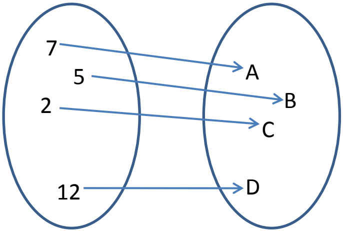
Surjection



Bijective Function

A function is **bijective** if it is both injective (one-to-one) and surjective (onto)

Bijection



Sometimes we call this a “one to one correspondence”

Consider the function $f(x) = sin(x)$

Which is true?

A. Both one-to-one and onto (bijection)
B. Neither one-to-one nor onto
C. One-to-one but not onto
D. Onto but not one-to-one
E. I don’t know

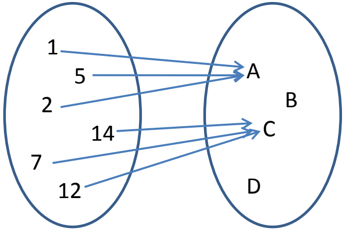
Inverse Image

Let *y* be an element of the co-domain of a function.

The **inverse image** of *y* is the subset of the domain that maps to *y*.

What is the inverse image of…

- A?
- C?
- B?



Inverse Function

Let f be a function. The **inverse** of f, denoted f^{-1}, is a function that "reverses" f.

e.g.: $f(7)$ = "aardvark" $\leftrightarrow$ $f^{-1}(\text{"aardvark"}) = 7$

Not all functions have inverses.

Suppose $f : A \to B$, and consider $f^{-1} : B \to A$

- What can we say about f^{-1} if f is not one-to-one?
- What can we say about f^{-1} if f is one-to-one but not onto?
- What can we say about f^{-1} if f is a bijection?

Proving (or disproving) that a function is Injective

Let $f: D \to C$ such that...

Claim: f is 1 to 1.
Proof:
Let $a, b \in D$ such that $f(a) = f(b)$.
...
a = b

Claim: f is not 1 to 1.
Proof:
[Find two different elements of the domain that are mapped to the same value]

Examples:

- $f: R \to R$ such that f(x) = 3x + 7
- $f: R - \{1\} \to R$ such that f(x) = (x+1)/(x-1)
- $f: Z \to Z$ such that f(x) = x mod 7

Proving (or disproving) that a function is Surjective

Let $f: D \rightarrow C$ such that…

Claim: f is onto.
Proof:
Let $c \in C$ (arbitrarily selected).
…
there exists $d \in D$ such that $f(d) = c$.

Claim: f is not onto.
Proof:
[Find an element of the codomain such for all $d \in D$, $f(d)$ is not equal to that element]

Examples:

- $f: R \rightarrow R$ such that f(x) = 3x + 7
- $f: R \rightarrow Z$ such that f(x) = $\lfloor x/2 \rfloor$
- $f: R^{+} \rightarrow R$ such that f(x) = $\sqrt{x}$

Proving a function is a bijection

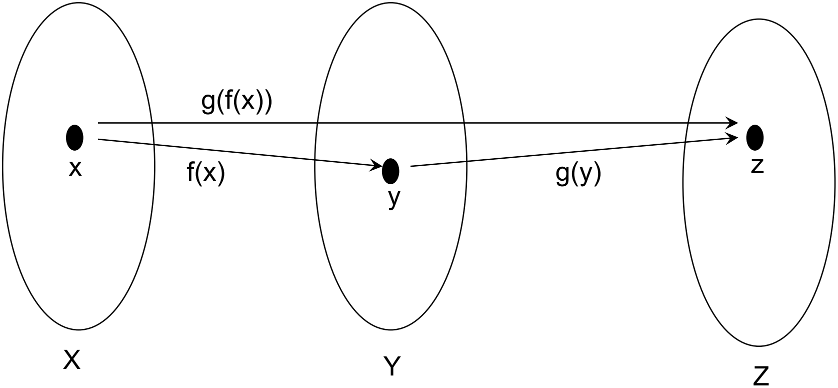
Let $f: D \rightarrow C$ such that…

Claim: f is a bijection.
Proof:
Part 1 [Prove f is one-to-one]…
Part 2 [Prove f is onto]…

Composition of functions

Let f: $X \to Y$ and g: $Y \to Z$

– $(g \quad f): X \to Z$ where $(\forall x \in X)[(g \quad f)(x) = g(f(x))]$



Composition of functions

Let f: X $\rightarrow$ Y   and g: Y $\rightarrow$ Z.

What can we say about the domain and range of
g   f ?

Composition on finite sets- example

- Example

  X = {1,2,3}, Y = {a,b,c,d,e}, Z = {x,y,z}

f(1) = c	g(a) = y	(g f)(1) = ?
f(2) = b	g(b) = y	(g f)(2) = ?
f(3) = a	g(c) = z	(g f)(3) = ?
	g(d) = x	
	g(e) = x	

Composition for infinite sets- example

$f: Z \rightarrow Z \quad f(n) = n + 1$

$g: Z \rightarrow Z \quad g(n) = n^2$

$(g \quad f)(n) = g(f(n)) = g(n+1) = (n+1)^2$

$(f \quad g)(n) = f(g(n)) = f(n^2) = n^2 + 1$

Note: $g \quad f \neq f \quad g$