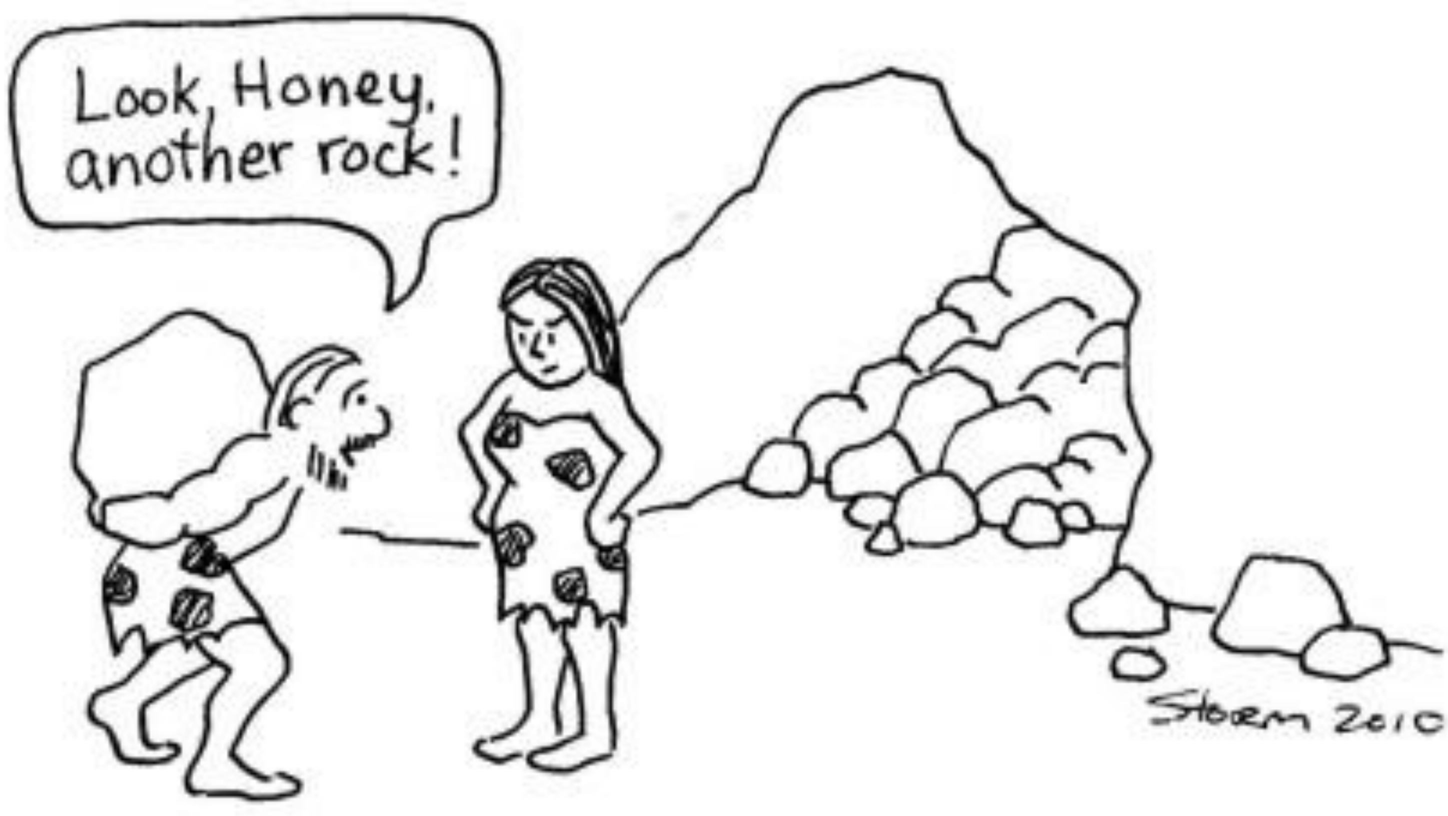
Cardinality and Infinity

Recall:
- Cardinality is the "size" of a set.
- Notation: $|A|$ is the cardinality of A.

Questions:
- Can one infinite set be "larger" than another?
- How can we compare the sizes of infinite sets?

Rocks

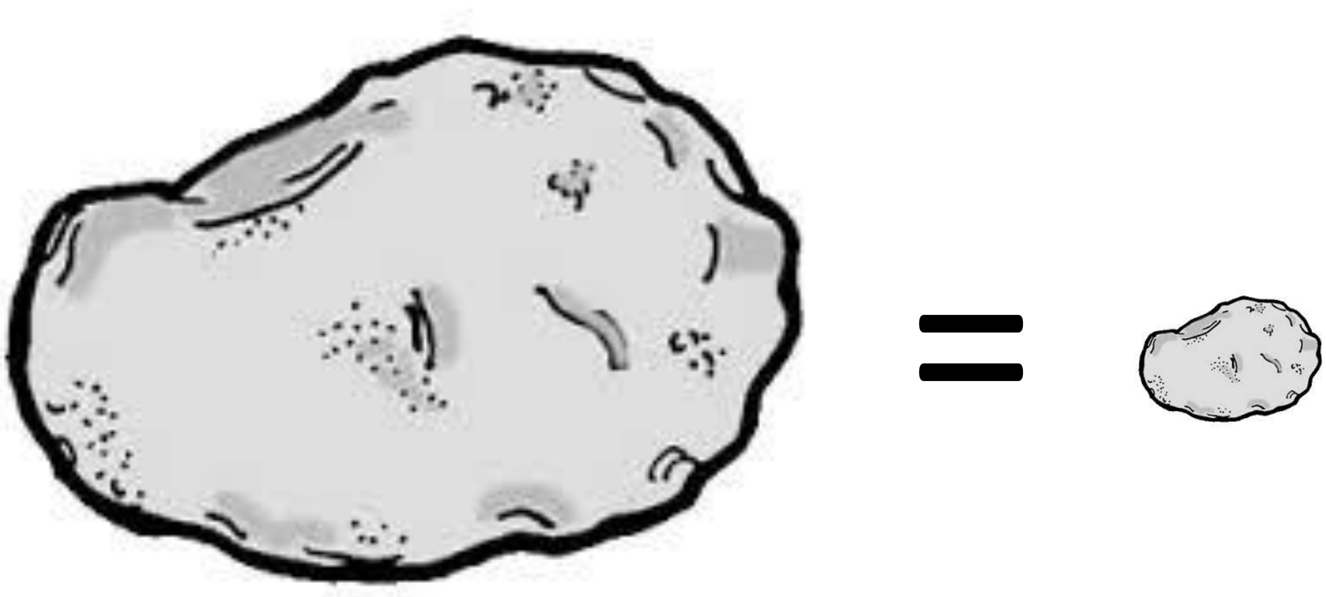
In the days of cave men, wealth was measured with rocks.*



*I'm pretty sure this statement is false.

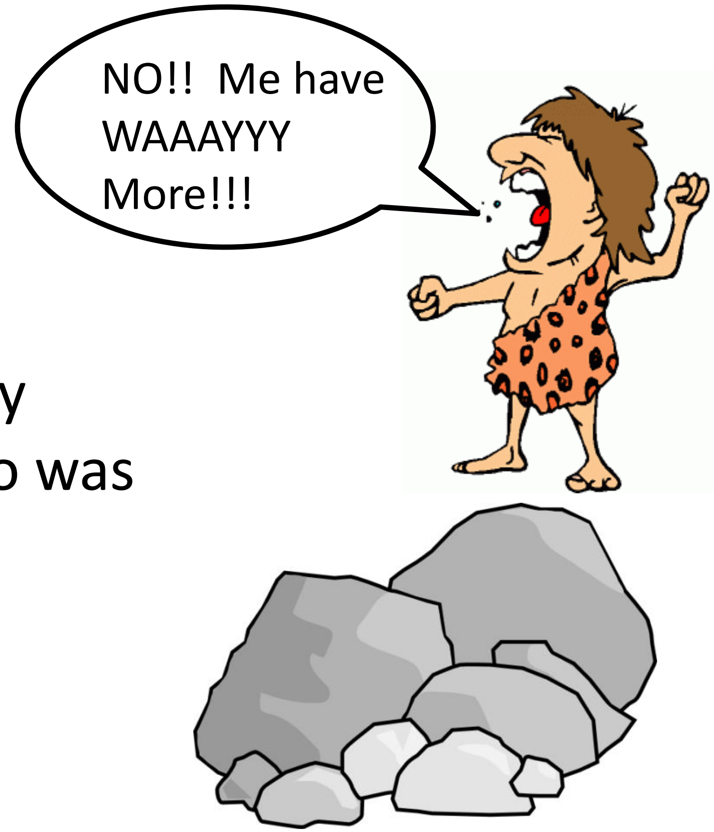
A rock is a rock

Each rock was worth “1 rock”. Cavemen weren’t good with measuring sizes.



Conflict Resolution

Cavemen often argued about who had accumulated more wealth. The problem was… they didn't know how to count yet.



How could they determine who was wealthier?

Comparing Cardinality

Idea #1: If we can find a one-to-one function mapping A to B, then $|A| \leq |B|$.

Idea #2: If we can find an onto function mapping A to B, then $|A| \geq |B|$.

Idea #3: If we can find a bijective function mapping A to B, then $|A| = |B|$.

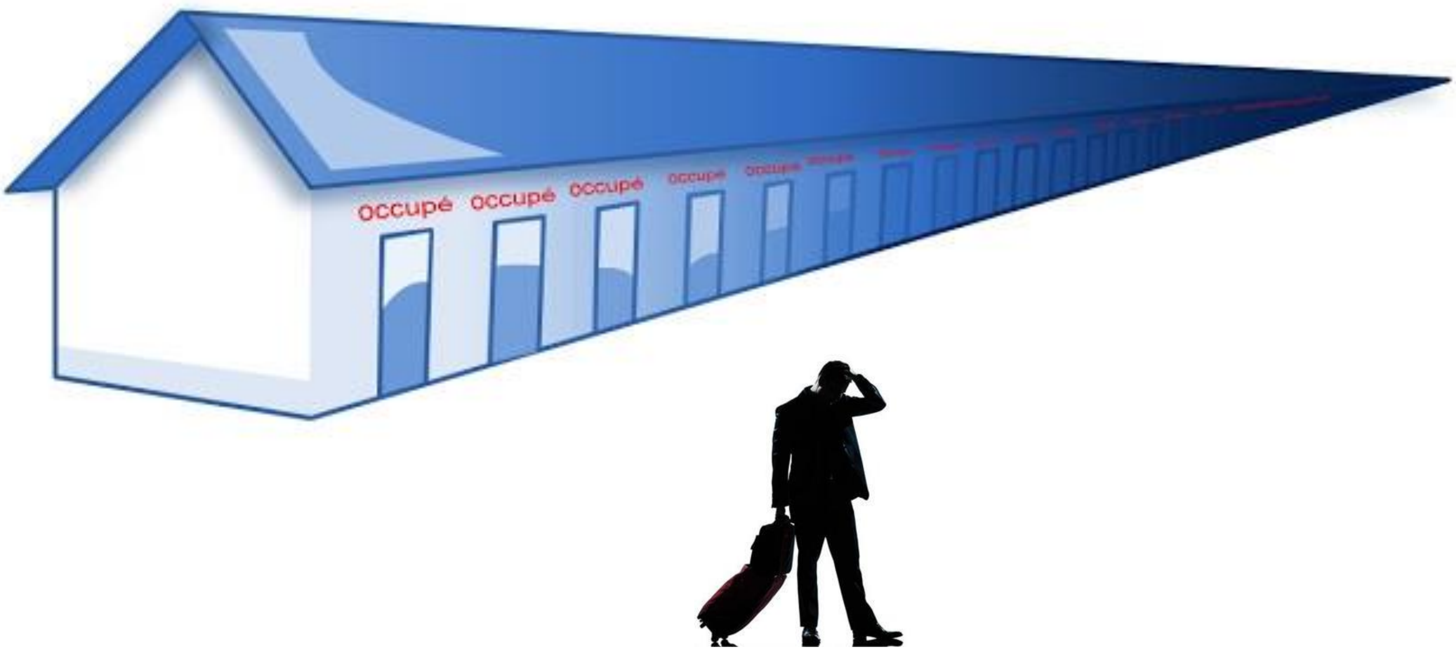
- These ideas allow us to compare sizes of sets without counting the elements, so we can apply them to infinite sets!

The Future is Grim

Overpopulation is a real problem. In fact… I foresee a day when infinitely many people will inhabit the Earth.*

*This statement is utterly ridiculous. Please don't believe it.

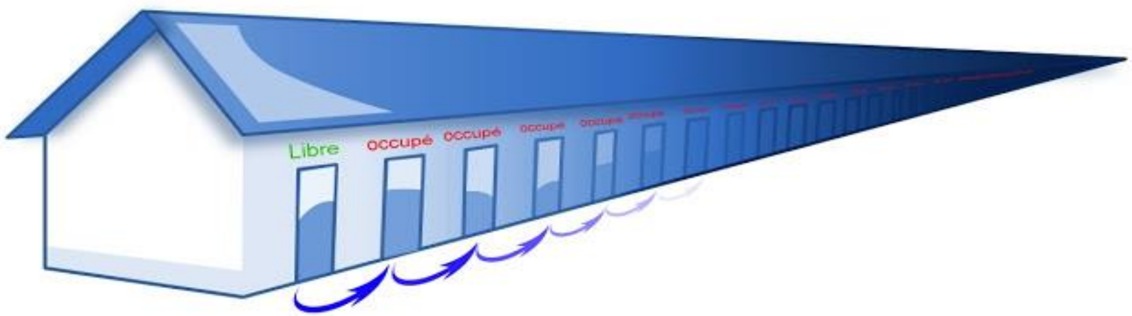
The Infinite Hotel



The hotel is full! What can we do?

The Infinite Hotel

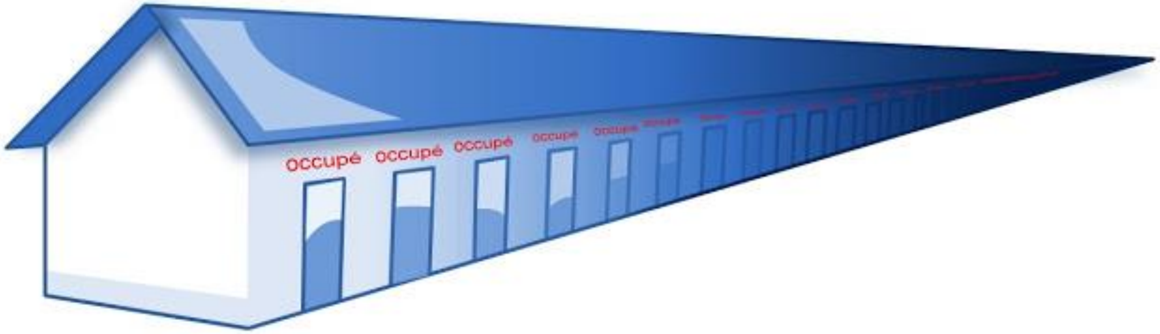
Solution:



Idea: Adding an element to an infinite collection does not make it any “bigger”.

If A is an infinite set, then: $|A \cup \{*\}| = |A|$

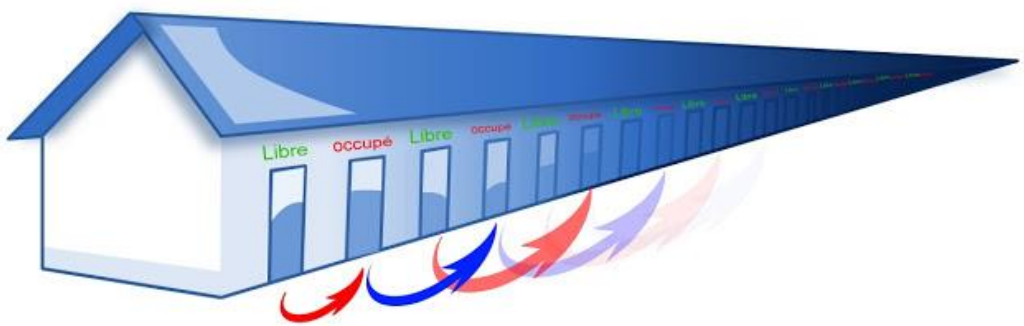
Can we demonstrate this idea with an injective function?



Worse Problem…

No problem.

Solution:



Idea: Adding an infinite collection to an “equally sized” infinite collection does not make it any bigger.

If A and B are infinite sets of the same “size” then:
$|A \cup B| = |A|$

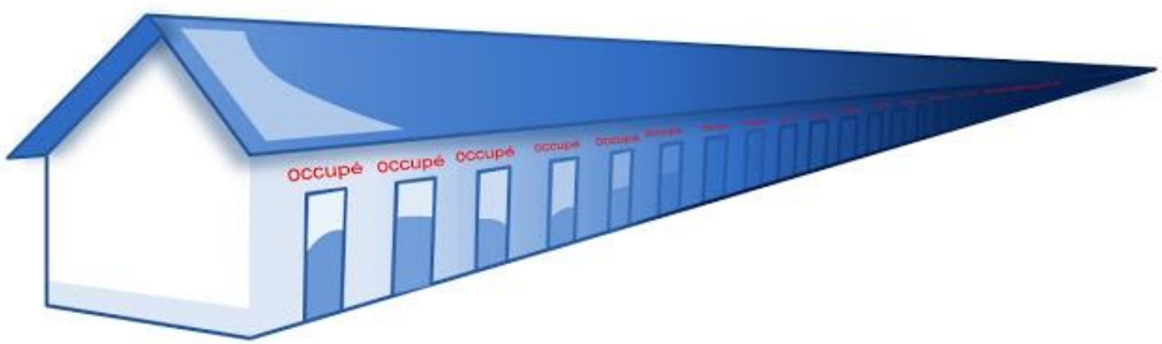
Can we demonstrate this idea with a injective function?

Nightmarishly bad problem…

In my town there are infinitely many infinite hotels (numbered 1, 2, 3…). Imagine they are all full of people.

All of the hotels burn down at the same time, but everyone survives.

All of these people arrive at MY infinite hotel needing a room…



Infinity is Weird

Idea: Infinitely many identically sized infinite collections (numbered 0, 1, 2…) can be merged together to form a collection that is no bigger than any one of those we started with.

Let A_0 be an infinite set, and let A_1, A_2, $A_{3\ldots}$ be an infinite list of sets, all the same size as A_0. Then:

$$|A_0 \cup A_1 \cup A_2 \cup A_3 \cup \ldots| = |A_0|$$

Let's demonstrate this with an injective function.

Countability

Fact: There is no infinite set that is smaller than $\mathbb{N}$.

An infinite set that has the same cardinality as $\mathbb{N}$ is said to be **countable**. Sets that are *bigger* than $\mathbb{N}$ are called **uncountable**.

How can we show a set is countable?
(Can you think of two ways?)

How can we show a set is uncountable?
(Can you think of two ways?)

Countability

- Is $\mathbb{Z}$ countable?
- Is $\mathbb{Q}$ countable?
- Is $\mathbb{R}$ countable?
- Is the set of all finite ASCII strings countable?