Unit 11
Relations

Relations

A **relation** (among sets) is a subset of their Cartesian product.

Relations can involve any number of sets, but frequently they are **binary** (two sets).

Examples of Binary Relations

Let S = {Students at Maryland}

Let F = {faculty members at Maryland}

Define relation R on $S \times F$ by:

$R = \{<x, y> \in S \times F$: x has been in a class taught by y}

Notation:

aRb means $<a,b> \in R$

Examples of Binary Relations

- Any predicate with two free variables (over fixed domains) defines a binary relation over the same domains:

  For all $x \in A$, $y \in B$

  xRy iff $P(x, y)$ is true

- $<$ is a binary relation on $\mathbb{R}\times\mathbb{R}$, or $\mathbb{Z}\times\mathbb{Z}$, etc.

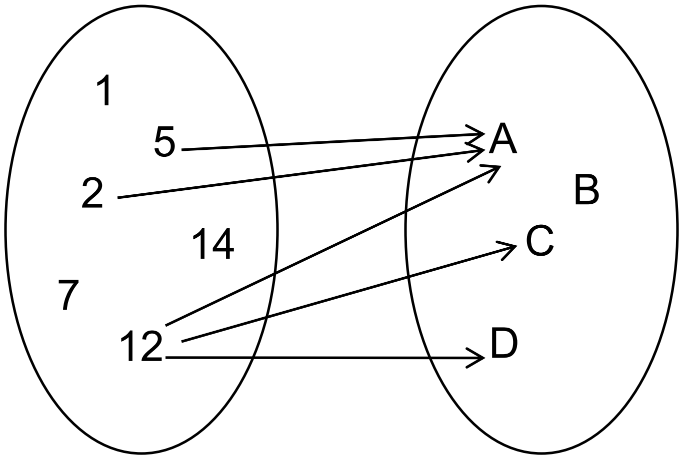
- Any function can be thought of as a binary relation

  (Can any binary relation be thought of as a function?)

- $=$ can be thought of as a (simple) binary relation over any domain

Ways to represent Binary Relations

- Arrow Diagrams

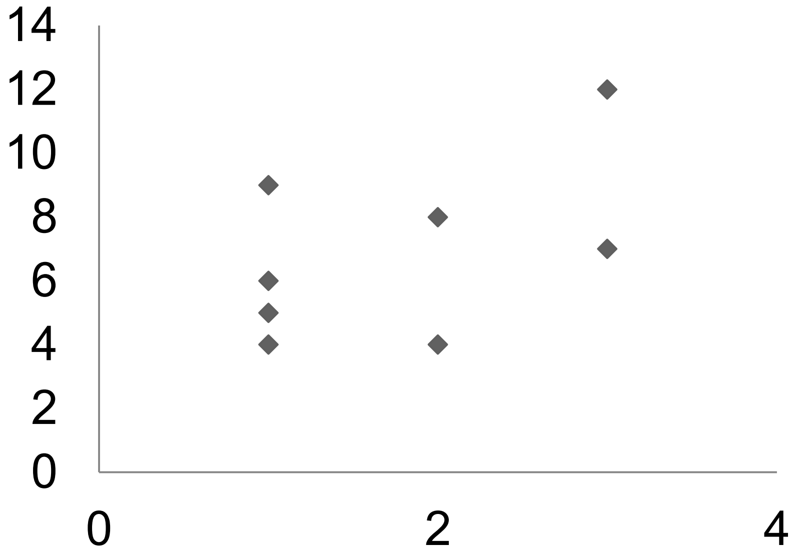


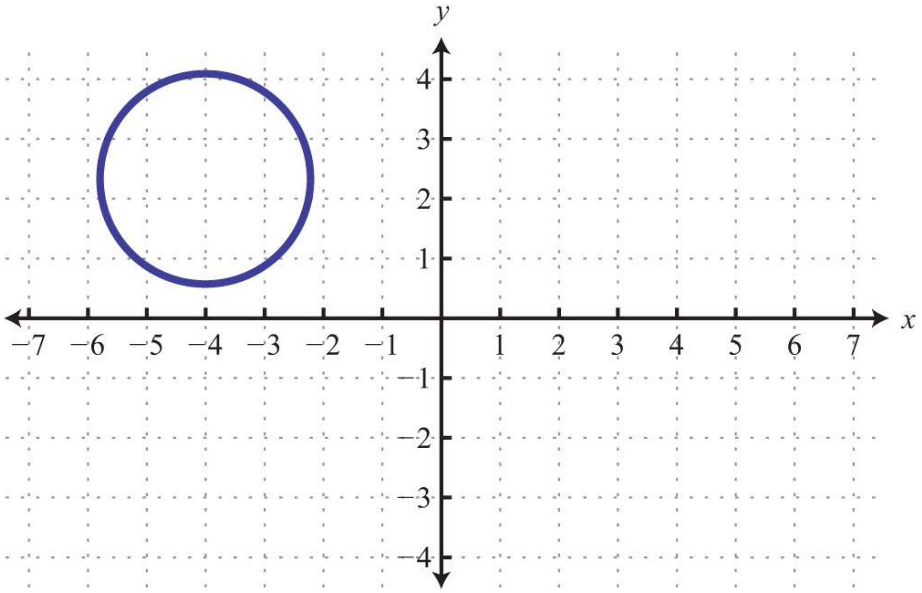
- Set Notation

R = {<5,A>, <2,A>, <12,A>, <12, C>, <12,D>}

Ways to represent Binary Relations

- Graphs





Ways to represent Binary Relations

- Matrix Representation

R = {(2,1),(3,1),(3,2)}   could also be represented as:

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array}$$

Ternary Relations

Examples:

- Let $R \subseteq \mathbb{Z}\times\mathbb{Z}\times\mathbb{N}$ be defined by:

  $\langle a, b, c\rangle \in R$ if and only if $a \equiv_c b$

  Alternate notation (like a predicate):

  $R(a, b, c)$ holds if and only if $a \equiv_c b$.

- Let $R \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ be defined by:

  $\langle a, b, c\rangle \in R$ iff there could be a triangle with sides of lengths a, b and c.

Unary Relations

What would a **Unary Relation** look like?

Examples?

n-ary Relations

Relations can involve any number of sets.

Example:

Let $n \in \mathbb{N}^+$

Define $R \subseteq \mathbb{R}^n$ $(\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\ldots\times\mathbb{R})$ as:

$\langle x_1, x_2, x_3, \ldots, x_n \rangle \in R$ if and only if $\sqrt{{x_1}^2 + {x_2}^2 + {x_3}^2 + \cdots + {x_n}^2} \leq 1$

What is the geometric interpretation for…

n=2?

n=3?

n=1?

n=4???

Properties of Binary Relations

Reflexive	$(\forall a \in A)\ [aRa]$
Irreflexive	$(\forall a \in A)\ [a \not R a]$
Symmetric	$(\forall a, b \in A)\ [aRb \rightarrow bRa]$
Antisymmetric	$(\forall a, b \in A)\ [aRb \wedge bRa \rightarrow a = b]$
Asymmetric	$(\forall a, b \in A)\ [aRb \rightarrow b \not R a]$
Non-symmetric	$(\forall a, b \in A)\ [a \neq b \rightarrow (aRb \leftrightarrow b \not R a)]$
Transitive	$(\forall a, b, c \in A)\ [aRb \wedge bRc \rightarrow aRc]$

Which Properties Hold?

Which of the properties on the previous slide hold for…

- $<$ over $\mathbb{R}$
- $=$ over the set {A, B, C}
- R over $\mathbb{N}$ such that aRb iff a is a factor of b
- R over $\mathbb{N}$ such that aRb iff $a \equiv_7 b$
- **R** over {students in this class} such that aRb iff a considers b to be a friend