CMSC 250 Homework #4, DUE: Thursday, 3/28

1. For each of the following statements, either prove the statement or give a counterexample that shows the statement is false. **We will use the (non-standard) notation $\mathbb{I}$ to represent the irrational numbers.**

   Each problem is worth 5 points.

   a. For all $m \in \mathbb{N}^{>2}$, $m^2 - 1$ is composite.
   b. For all integers m, if 7 is a factor of m then 7 is not a factor of $m + 6$.
   c. $(\forall x \in \mathbb{I}^+)[\sqrt{x} \in \mathbb{I}]$
   d. $(\forall x,\, y \in \mathbb{Q})(\forall z \in \mathbb{I})[\text{If } y \neq 0 \text{ then } x + yz \in \mathbb{I}]$
   e. $log_5(2) \in \mathbb{I}$. Hint: Consider using the Fundamental Theorem of Arithmetic.

2. (5 points) What is the remainder when 7^{5555} is divided by 6? Explain briefly.

3. (5 points) What is the remainder when 5^{7777} is divided by 6? Explain briefly.

4. (5 points) What is the remainder when 5^{6666} is divided by 6? Explain briefly.

5. (5 points) Let k be the number of people on the planet Earth. What is the remainder when $15k^{200} + 6(k+2)^{71} + 302$ is divided by 3? Explain briefly.

6. (5 points) Use modular congruence (mod 2) to decide whether or not the following number is even or odd:
   $722^{77} - 333^{99}(55^{100})$.

7. (10 points) An "Equivalence Relation" is reflexive, symmetric, and transitive. Prove that modular congruence is an equivalence relation, by proving the following:

   a. Prove that the modular congruence is reflexive: $\forall x \in \mathbb{Z}, \forall n \geq 1[x \equiv_n x]$
   b. Prove that modular congruence is symmetric: $\forall x, y \in \mathbb{Z}, \forall n \geq 1[x \equiv_n y \rightarrow y \equiv_n x]$
   c. Prove that modular congruence is transitive: $\forall x, y, z \in \mathbb{Z}, \forall n \geq 1[(x \equiv_n y \text{ and } y \equiv_n z) \rightarrow x \equiv_n z]$

8. (10 points) Prove that for all natural numbers, n: n is not congruent to $n^2 - 4$ (mod 9)

9. (10 points) Show that if n is a natural number and n is congruent to 3 (mod 4) then one of the prime factors of n must also be congruent to 3 (mod 4).

10. (10 points) Prove that for all integers n and d: $d|n \leftrightarrow n = d.\lfloor \frac{n}{d} \rfloor$

11. (10 points) Prove that for any real number x: If x is not an integer, then $\lfloor x \rfloor + \lfloor -x \rfloor = -1$