Announcements

- HW #1 is due on Wednesday 2/13 at 11:00PM
- Note: Opinions are NOT statements!

Unit 3
Predicate Logic

Propositional Logic falls short

All men are mortal
Socrates is a man
∴ Socrates is mortal

In mathematics we need to express:

- A property is true about a particular element
- A property is true for all elements in a set
- A property is true for at least one element in a set

Predicates

A **predicate** is a sentence (containing variables) that is either true or false depending on what values are substituted for the variables. The domain for the variables must be specified when the predicate is defined.

We use notation like “P(x)” or “Q(x, y)” to denote predicates.

Examples:

- P(x) = “x is even”, where $x \in \mathbb{Z}$
- R(a) = “a is rational”, where $a \in \mathbb{R}$
- G(y) = “y is green”, where y is an M&M
- F(a, b) = “a is a factor of b”, where $a, b \in \mathbb{N}$
- T(a, b) = “a is taller than b”, where a, b are students in this class
- P(x, y, z) = “$x^2 + y^2 = z^2$”, where $x, y, z \in \mathbb{R}$

Logical Connectives

The logical connectives can be used to join predicates to make more complex predicates:

- $P(x) = {\sim}Q(x) \vee R(x)$
- $T(x, y) = (A(x) \wedge G(x, y)) \rightarrow {\sim}L(y)$

Universal Quantifier: $\forall$

$(\forall x \in D)\ [P(x)]$

"For all values x in the set D, property P is true about x."
"For all x in D, P(x) holds."
"P holds for every element of D"

Examples (Are these true?):

- Let E(x) = "x is even", $x \in \mathbb{Z}$
  $(\forall x \in \mathbb{Z})\ [E(x) \vee E(x + 1)]$
- Let P(x) = "x is prime", $x \in \mathbb{N}$
  $(\forall x \in \mathbb{N})\ [P(x) \rightarrow P(x + 2))]$

Existential Quantifier: $\exists$

$(\exists x \in D)\ [P(x)]$

“There exists (at least one) value x in the set D such that property P is true about x”

“There exists x in D such that P(x) holds.”

“P holds for some element in D.”

Examples (Are these true?):

- Let N(x) = “x is negative”, $x \in \mathbb{Z}$; let P(x) = “x is positive”, $x \in \mathbb{Z}$
  $(\exists x \in \mathbb{Z})\ [N(x) \wedge P(x)]$
  $(\exists x \in \mathbb{Z})\ [N(x)] \wedge (\exists y \in \mathbb{Z})\ [P(y)]$

- Let G(x, y) = “x is greater than y”, $x,y \in \mathbb{N}$
  $(\exists x \in \mathbb{N})\ [G(x, x^2)]$

- Let G(x, y) = “x is greater than y”, $x,y \in \mathbb{R}$
  $(\exists x \in \mathbb{R})\ [G(x, x^2)]$

Specifying Domains

Sometimes we don't write the domain for quantified variables explicitly:

$(\exists x)\ [P(x)]$
$(\forall x)\ [S(x) \vee Q(x)]$

Possible reasons:

- Domain is specified in advance
- Domain may be obvious from context

Negating

Practice translating into statements of Predicate Logic:

- There are no honest politicians
- Not all books are interesting

Now Let's discuss cats…

What is the opposite of "All cats are lazy"?

$\sim((\forall x)\ [L(x)]) \equiv (\exists x)\ [\sim L(x)]$

What is the opposite of "There is a cat who can fly"?

$\sim((\exists x)\ [F(x)]) \equiv (\forall x)\ [\sim F(x)]$

Vacuous Statements

Is it possible for **both** of the following to be true?

$(\forall x \in D)\ [Q(x)]$
$(\forall x \in D)\ [\sim Q(x)]$

- If D is empty, then $(\forall x \in D)[P(x)]$ is vacuously true.
- If D is empty, then $(\exists x \in D)[P(x)]$ is vacuously false.

Prelude to Proofs…

How can we prove or disprove quantified statements?

- Showing Existential statement is true: Usually easy. Why?
- Showing Universal statement is false: Usually easy. Why?
- Showing Existential statement is false: Frequently hard. Why?
- Showing Universal statement is true: Frequently hard. Why?

- Intuitive Examples

Multiple Quantifiers

Let P = {four student volunteers}
Let C = {three chairs}
Let S(x, y) = “Person x is sitting in chair y”

What is the meaning of the following statements?
(Let’s demonstrate interpretations where each of these are true/false)

1. $(\exists p \in P)\ (\exists c \in C)\ [S(p,c)]$
   $(\exists c \in C)\ (\exists p \in P)\ [S(p,c)]$
2. $(\forall p \in P)\ (\forall c \in C)\ [S(p,c)]$
   $(\forall c \in C)\ (\forall p \in P)\ [S(p,c)]$
3. $(\exists p \in P)\ (\forall c \in C)\ [S(p,c)]$
4. $(\forall c \in C)\ (\exists p \in P)\ [S(p,c)]$
5. $(\forall p \in P)\ (\exists c \in C)\ [S(p,c)]$
6. $(\exists c \in C)\ (\forall p \in P)\ [S(p,c)]$

Negations of multiply quantified statements

- $\sim((\forall x)\ (\exists y)\ [Q(x,y)])\ \equiv\ ?$
- $\sim((\exists a)\ (\forall b)\ (\forall c)(\exists d)\ [R(a,b,c,d)])\equiv\ ?$

Notation for Repeated Quantifiers

Instead of writing:

$(\exists x \in D)\ (\exists y \in D)\ (\exists z \in D)\ [P(x,y,z)]$

We may write:

$(\exists x,y,z \in D)\ [P(x,y,z)]$

This also applies to multiple universal quantifiers.

Practice Translating

For these examples, the domain is natural numbers ($\mathbb{N}$). We may use quantifiers, variables, the symbols +, -, ×, ÷, =, <, >, and constants like 1 or 57.

Predicates:

- x is even
- x is odd
- x is a square
- x is composite