Announcements

• Did you assign page numbers to each question?
• Homework #2 has been posted
Writing Proofs

A good proof should have:

– a statement of what is to be proven
– "Proof:" to indicate where the proof starts
– Step by step derivation starting with the premises and ending with the desired conclusion.
– a clear indication of the reason for each step
– careful notation, completeness and order
Statement of Claims

The following are equivalent:

- Every even number (greater than 3) is the sum of two primes.
- For all $n \in \mathbb{N}^{\text{Even}}$: If $n>3$, then $n$ is the sum of two primes.
- $(\forall n \in \mathbb{N}^{\text{Even}})[n>3 \rightarrow (\exists a, b \in \mathbb{N}^{\text{Prime}})[n=a+b]]$
- $(\forall n \in \mathbb{N})[(\exists k \in \mathbb{N})[n=2k] \land n>3 \rightarrow (\exists a, b \in \mathbb{N})[a>1 \land (\forall c, d \in \mathbb{N})[cd=a \rightarrow c=1 \lor d=1] \land b>1 \land (\forall e, f \in \mathbb{N})[ef=b \rightarrow e=1 \lor f=1] \land n=a+b]]$

Which of these would you use to state your claim?
Constructive Proofs of Existence

• Claim: \((\exists a, b \in \mathbb{N})(a^b = b^a \land a \neq b)\)

• Claim: There exist three natural numbers, a, b, and c (all distinct) such that \(a^2 + b^2 = c^2\)

• Claim: 23 can be written as the sum of 9 cubes (of non-negative integers).

• Claim: There is a number that can be written as the sum of two cubes (of positive integers) in two different ways.

• Talk about “Taxicab” numbers, and “non-constructive” proofs of existence.
Proofs by Exhaustion/Cases

• Claim: \( \forall n \in \{1, 2, 3, 4\} \ [ (n + 1)^3 \geq 3^n] \).

• Claim: There are no integer solutions to the equation \( a^2 + b^2 = 7 \)

• Claim: 23 cannot be written as the sum of 8 cubes (of non-negative integers).

• Mention proof of four-color problem
Applying Universal Generalization

- The most common technique for proving *universally quantified* statements.
- If you’re not sure how to start – try this!

\[
\text{Claim: } \left( \forall x \in D \right)[P(x)]
\]

**Proof:**

Let \( d \in D \), *arbitrarily chosen*.

... 

\( P(d) \)

Since \( d \) was chosen arbitrarily, \( P(x) \) holds for all \( x \in D \).
Example of Proving a Universal Statement

Claim: \((\forall n \in \mathbb{N}_{\text{Even}})[n^2 \text{ is even}])\)
Proof Example: Rigid Style

Claim: \((\forall n \in \mathbb{N}^{\text{Even}})[n^2 \text{ is even}]\)

Proof:

1. Let \(a \in \mathbb{N}^{\text{Even}}\), selected arbitrarily
2. \(a = 2k\), for some \(k \in \mathbb{N}\) \[Defn \text{ of “even”}\]
3. \(a^2 = (2k)(2k)\)
4. \(a^2 = 2(2k^2)\)
5. (Note that \(2k^2 \in \mathbb{N}\)) \[Since \mathbb{N} \text{ is closed under multiplication}\]
6. \(a^2 \text{ is even} \[Defn \text{ of “even”; using (4), (5)\]}
7. Since \(a\) was chosen arbitrarily, \((\forall n \in \mathbb{N}^{\text{Even}})[n^2 \text{ is even}]\)
Proof Example: Flowing Style

**Claim:** The square of any even natural number is even.

**Proof:**

Let \( a \in \mathbb{N}^{\text{Even}} \), selected arbitrarily. Since \( a \) is even, \( a = 2k \) for some \( k \in \mathbb{N} \). Squaring both sides, we get \( a^2 = (2k)^2 = 2(2k^2) \). Noting that \( 2k^2 \in \mathbb{N} \) (\( \mathbb{N} \) is closed under multiplication), we see that \( a^2 \) is equal to twice a natural number, hence \( a^2 \) is even. Since \( a \) was selected arbitrarily, the proposition holds for any even number.

• Which style do you think is easier to understand?
• Which style is easier to write without making mistakes?
• Can we use a style that is somewhere between these two?
More Examples

• Claim: The product of two odd integers is odd.
• Claim: \( \mathbb{Q} \) is closed under multiplication. (Assuming we know that \( \mathbb{Z} \) is closed under multiplication.)
• Claim: \((\forall n \in \mathbb{N}^{>0}), n^2 + 3n + 2\) is composite.