Announcements

- Homework #6 due tomorrow

Constructive induction

$$a_0 = 2 \qquad a_1 = 7$$

$$(\forall k \in Z^{\geq 2})\left[a_k = 12a_{k-1} + 3a_{k-2}\right]$$

We suspect that this recurrence is bounded by some exponential function of the form AB^n, where A and B are integers:

$$(\forall n \in Z^{\geq 0})\left[a_n \leq A \cdot B^n\right]$$

We would like to find the *smallest* integers A and B that make this work.

Structural Induction

Allows Induction on domains other than *N*.

Can be done on any structure that is defined *recursively*.

Example: Trees

Base: Prove Proposition is true about Tree of size 1
Inductive Hypothesis: Assume Proposition holds for a collection of sub-trees.
Inductive Step: Prove that the proposition must then hold for a new tree made by joining these subtrees as children of a new node.

Structural Induction

Definition: A k-ary tree is a tree where each node has at most k children.

Claim: The number of nodes in a k-ary tree is no more than

$$\frac{k^h - 1}{k - 1}$$ (h is the height of the tree)

Unit 8
Introduction to Set Theory

- Today: Lots (and lots) of definitions

- Next time: Begin doing proofs with sets

Set definitions

Definition of a set:

name of set = {list of elements, or a description of the elements}

Examples: $A = \{1,2,3\}$ or $B = \{x \in Z \mid -4 < x < 4\}$ or
$C = \{x \in Z^+ \mid -4 < x < 4\}$

A set is completely defined by its elements, i.e.,

$\{a,b\} = \{b,a\} = \{a,b,a\} = \{a,a,a,b,b,b\}$

More set concepts

- The universal set (U) is the set consisting of all possible elements in some particular situation under consideration
- A set can be finite or can be infinite
- For a set S, n(S) or |S| are used to refer to the cardinality of S, which is the number of elements in S
- The symbol $\in$ means "is an element of"
- The symbol $\notin$ means "is not an element of"

Subset

- $A \subseteq B \leftrightarrow (\forall x \in U)[x \in A \rightarrow x \in B]$

  A is contained in B

  B contains A

- $A \not\subseteq B \leftrightarrow (\exists x \in U)[x \in A \wedge x \notin B]$

- Relationship between membership and subset:

  $(\forall x \in U)[x \in A \leftrightarrow \{x\} \subseteq A]$

- Definition of set equality: $A = B \leftrightarrow A \subseteq B \wedge B \subseteq A$

Do these represent the same sets or not?

$X = \{x \in \mathbf{Z} \mid (\exists p \in \mathbf{Z})[x = 2p]\}$

$Y = \{y \in \mathbf{Z} \mid (\exists q \in \mathbf{Z})[y = 2q - 2]\}$

$A = \{x \in \mathbf{Z} \mid (\exists i \in \mathbf{Z})[x = 2i + 1]\}$

$B = \{x \in \mathbf{Z} \mid (\exists i \in \mathbf{Z})[x = 3i + 1]\}$

$C = \{x \in \mathbf{Z} \mid (\exists i \in \mathbf{Z})[x = 4i + 1]\}$

Formal definitions of set operations

Union: $$A \bigcup B = \{x \in U \mid x \in A \vee x \in B\}$$

Intersection: $$A \bigcap B = \{x \in U \mid x \in A \wedge x \in B\}$$

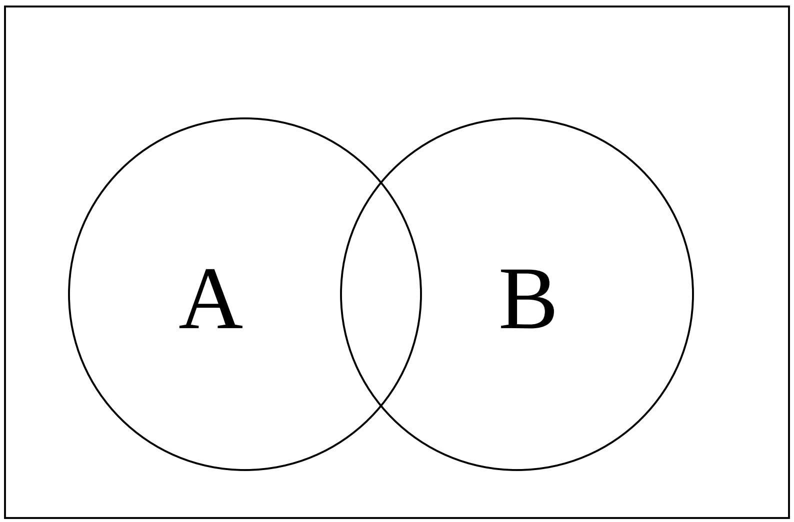
Complement: $$A^c = A' = \overline{A} = \{x \in U \mid x \notin A\}$$

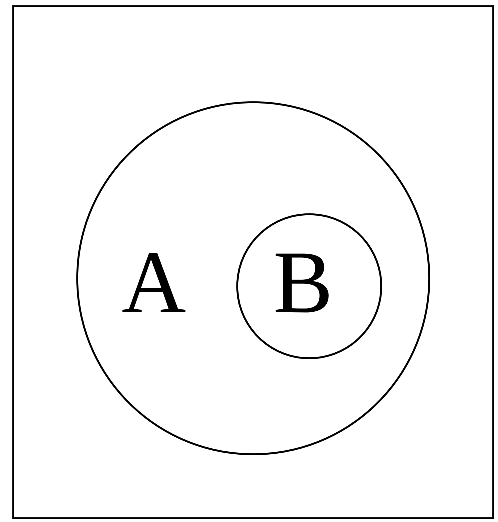
Difference: $$A - B = \{x \in U \mid x \in A \wedge x \notin B\}$$

$$A - B = A \bigcap B'$$

Venn diagrams

Sets are represented as regions (usually circles) in the plane in order to graphically illustrate relationships between them.





- Practice identifying union, intersection, difference compliment
- Can we draw Venn diagrams with more than 2 sets?

The empty set and its properties

The empty set $\varnothing$ has no elements, so $\varnothing = \{\}$.

1. $(\forall \text{ sets } X)[\varnothing \subseteq X]$ (Why?)
2. There is only one empty set. (Why?)
3. $(\forall \text{ sets } X)[X \cup \varnothing = X]$
4. $(\forall \text{ sets } X)[X \cap X' = \varnothing]$
5. $(\forall \text{ sets } X)[X \cap \varnothing = \varnothing]$
6. $U' = \varnothing$
7. $\varnothing' = U$

Ordered n-tuples

- An ordered n-tuple takes order and multiplicity into account
- The tuple $(x_1,x_2,x_3,\ldots,x_n)$
  - has n values
  - which are not necessarily distinct
  - and which appear in the order listed
- $(x_1,x_2,x_3,\ldots,x_n) = (y_1,y_2,y_3,\ldots,y_n) \leftrightarrow (\forall i \in 1 \leq i \leq n)[x_i = y_i]$
- 2-tuples are called pairs, and 3-tuples are called triples

The Cartesian product

- The Cartesian product of sets *A* and *B* is defined as

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

- $n(A \times B) = n(A) * n(B)$

Proper subset

$$A \subset B \leftrightarrow A \subseteq B \wedge A \neq B$$

Disjoint sets

A and B are disjoint

$\leftrightarrow$ A and B have no elements in common

$\leftrightarrow (\forall x \in U)[x \in A \rightarrow x \notin B \;\wedge\; x \in B \rightarrow x \notin A]$

$A \cap B = \varnothing \;\leftrightarrow$ A and B are disjoint sets