Announcements

- Homework #9 is due tomorrow.
- Exam #2 is on Monday 4/29.

Birthday Question (Revisited…)

How many people do you need to have in a room so that it is more than 50% likely that some pair of people in the room have the same birthday?

What would Jerry say? ☺

- Solution #1 from last time (an approximation that was easy to compute). What was wrong?

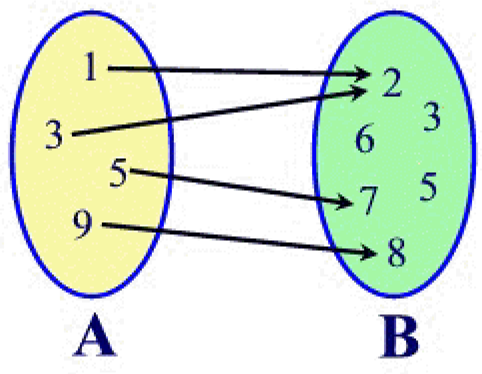
- Solution #2 (an exact answer)
  - Start with: Calculate probability that 4 people share no common birthday
    - How many ways to assign birthdays to a line of 4 people?
    - How many ways to assign DIFFERENT birthdays to a line of 4 people?
  - Generalize to n people

Unit 10
Functions

Function

- A **function** assigns members of one set (the **domain**) to members of another set (**co-domain**)
- The **range** is the subset of the co-domain that gets “hit”

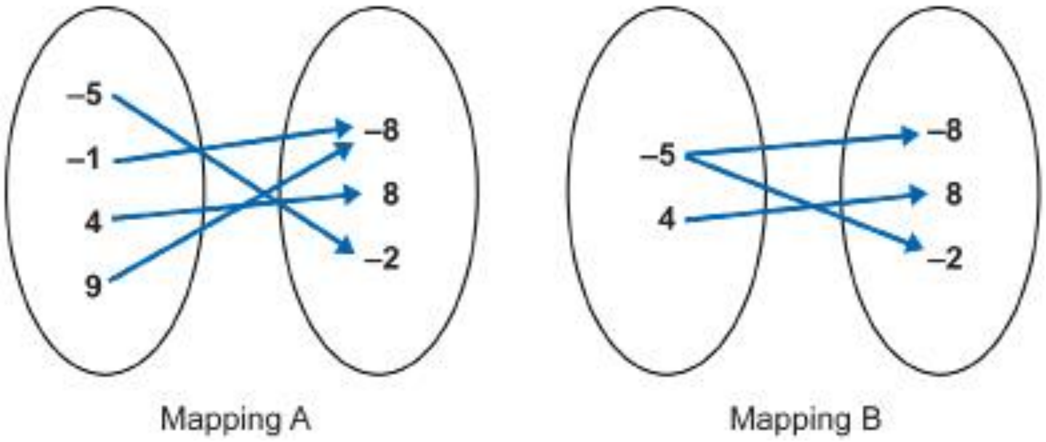
$f : A \rightarrow B$



Function

- A member of the domain can only be assigned to *one* member of the co-domain

Are these functions?



Function

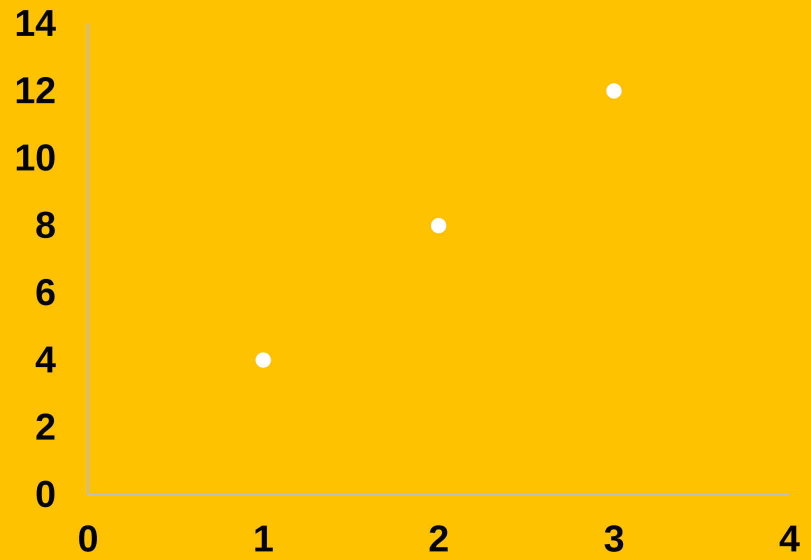
There are many ways to express functions:

Let $A = \{1, 2, 3\}$ $B=\{4, 8, 12\}$

$f : A \rightarrow B$ such that for all $a \in A$, $f(a) = 4a$

$f : A \rightarrow B$ such that $a \mapsto 4a$

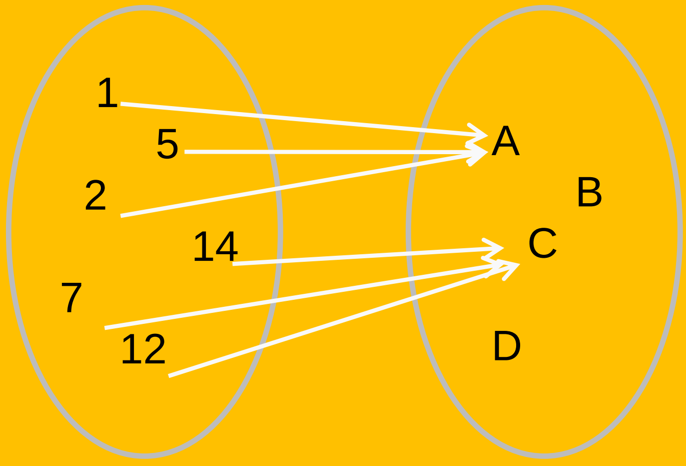
$f = \{<1, 4>, <2, 8>, <3, 12>\}$



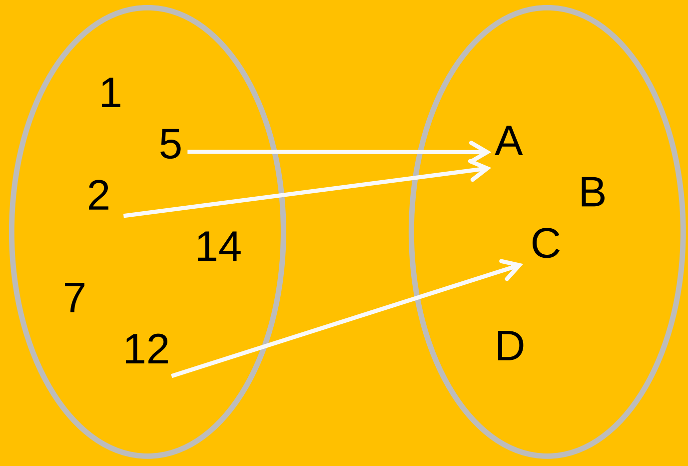
Total vs. Partial

- A **total function** assigns *every* member of the domain to an element of the co-domain
- A **partial function** may not assign every member of the domain

Total Function



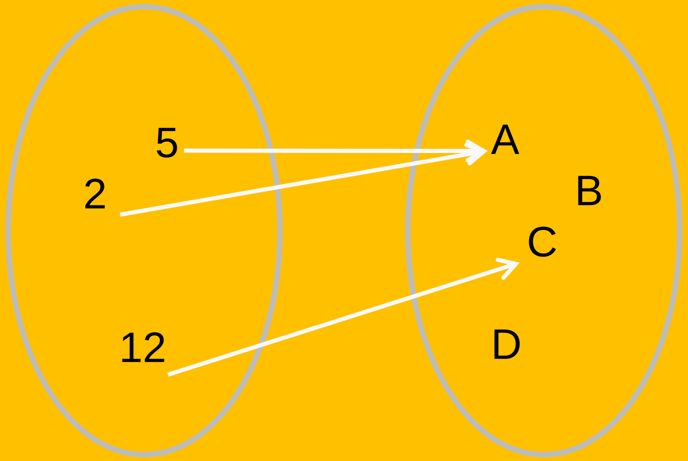
Partial Function



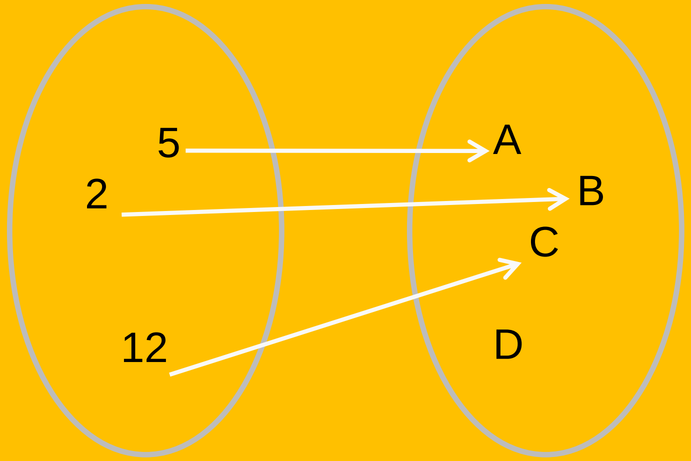
Injective Function

A function is **one-to-one** or **injective** if every element of the range is associated with *exactly one* element from the domain.

Not an Injection



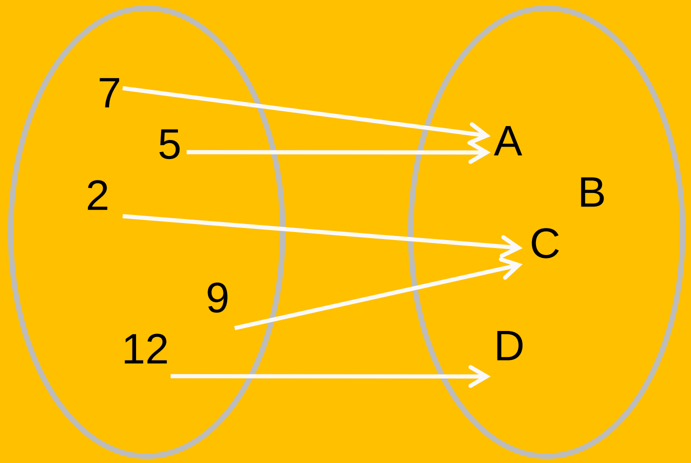
Injection



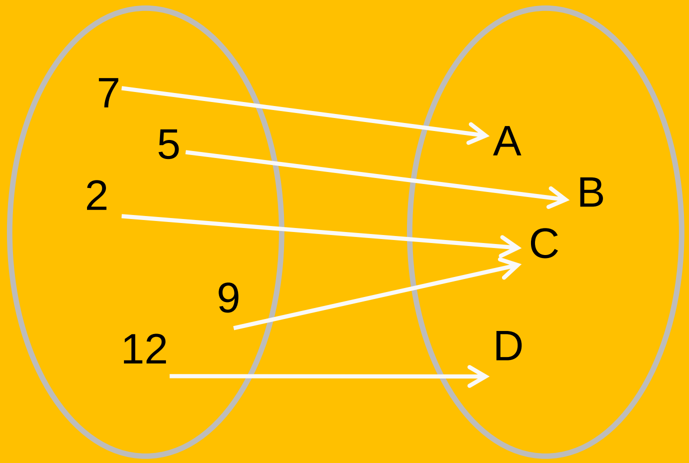
Surjective Function

A function is **onto** or **surjective** if the range is equal to the entire co-domain.

Not a Surjection



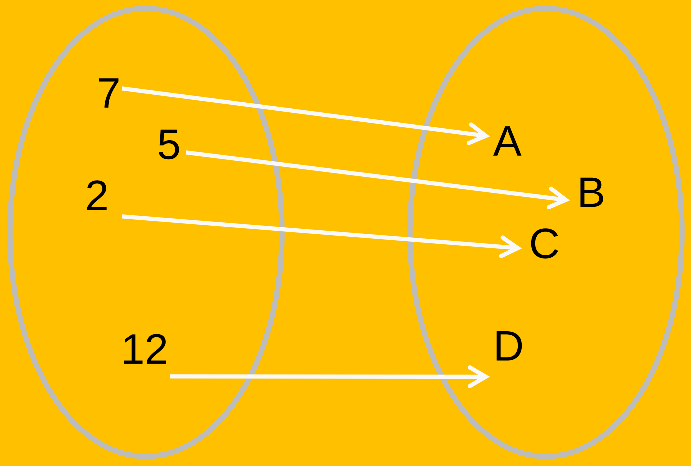
Surjection



Bijective Function

A function is **bijective** if it is both injective (one-to-one) and surjective (onto)

Bijection



Sometimes we call this a "one to one correspondence"

Let's vote… (No discussion please!)

Consider the function *f(x) = sin(x)*

Which is true?

A. Both one-to-one and onto (bijection)
B. Neither one-to-one nor onto
C. One-to-one but not onto
D. Onto but not one-to-one
E. I don't know

Answer: E

Domain and Co-Domain matter!

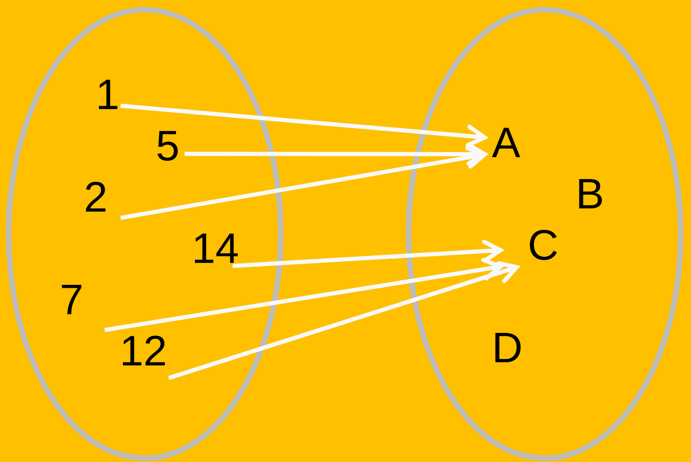
Inverse Image

Let *y* be an element of the co-domain of a function.

The **inverse image** of *y* is the subset of the domain that maps to *y*.

What is the inverse image of…

- A?
- C?
- B?



Inverse Function

Let f be a function. The **inverse** of f, denoted f^{-1}, is a function that "reverses" f.

e.g.: $f(7) =$ "aardvark" $\leftrightarrow$ $f^{-1}(\text{"aardvark"}) = 7$

Not all functions have inverses.

Suppose $f : A \rightarrow B$, and consider $f^{-1} : B \rightarrow A$

- What can we say about f^{-1} if f is not one-to-one?
- What can we say about f^{-1} if f is one-to-one but not onto?
- What can we say about f^{-1} if f is a bijection?

Proving (or disproving) that a function is Injective

Let $f: D \to C$ such that…

Claim: f is 1 to 1.
Proof:
Let $a, b \in D$ such that $f(a) = f(b)$.
…
a = b

Claim: f is not 1 to 1.
Proof:
[Find two different elements of the domain that are mapped to the same value]

Examples:

- $f: R \to R$ such that f(x) = 3x + 7
- $f: R - \{1\} \to R$ such that f(x) = (x+1)/(x-1)
- $f: Z \to Z$ such that f(x) = x mod 7

Proving (or disproving) that a function is Surjective

Let $f: D \rightarrow C$ such that…

<u>Claim: f is onto.</u>
<u>Proof:</u>

Let $c \in C$ (arbitrarily selected).
…
there exists $d \in D$ such that $f(d) = c$.

<u>Claim: f is not onto.</u>
<u>Proof:</u>

[Find an element of the codomain such for all $d \in D$, $f(d)$ is not equal to that element]

Examples:

- $f: R \rightarrow R$ such that f(x) = 3x + 7
- $f: R \rightarrow \mathbb{Z}$ such that f(x) = $\lfloor x/2 \rfloor$
- $f: R^{+} \rightarrow R$ such that f(x) = $\sqrt{x}$

Proving a function is a bijection

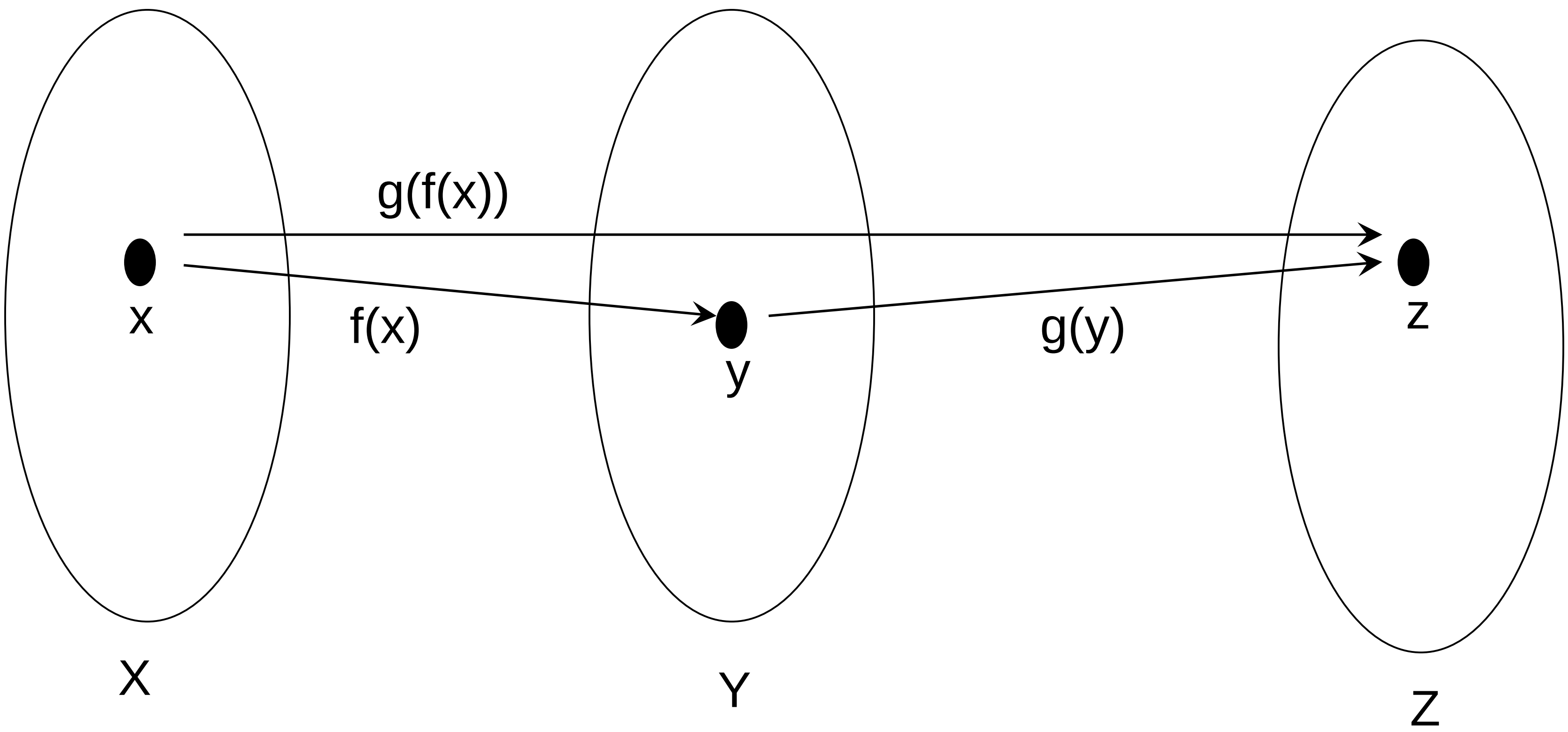
Let $f: D \to C$ such that…

Claim: f is a bijection.
Proof:
Part 1 [Prove f is one-to-one]…
Part 2 [Prove f is onto]…

Composition of functions

Let $f: X \to Y$ and $g: Y \to Z$

– $(g \circ f): X \to Z$ where $(\forall x \in X)[(g \circ f)(x)=g(f(x))]$



Composition of functions

Let f: $X \rightarrow Y$ and g: $Y \rightarrow Z$.

What can we say about the domain and range of $g \circ f$?

Composition on finite sets- example

- Example

  $X = \{1,2,3\}$, $Y = \{a,b,c,d,e\}$, $Z = \{x,y,z\}$

$f(1) = c$	$g(a) = y$	$(g \circ f)(1) = ?$
$f(2) = b$	$g(b) = y$	$(g \circ f)(2) = ?$
$f(3) = a$	$g(c) = z$	$(g \circ f)(3) = ?$
	$g(d) = x$	
	$g(e) = x$	

Composition for infinite sets- example

$f: Z \rightarrow Z \quad f(n) = n + 1$

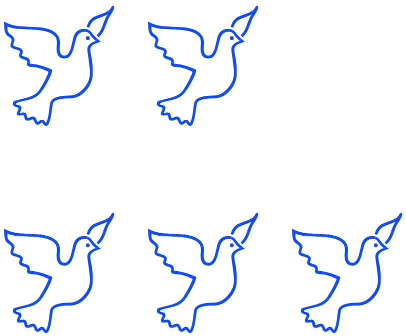
$g: Z \rightarrow Z \quad g(n) = n^2$

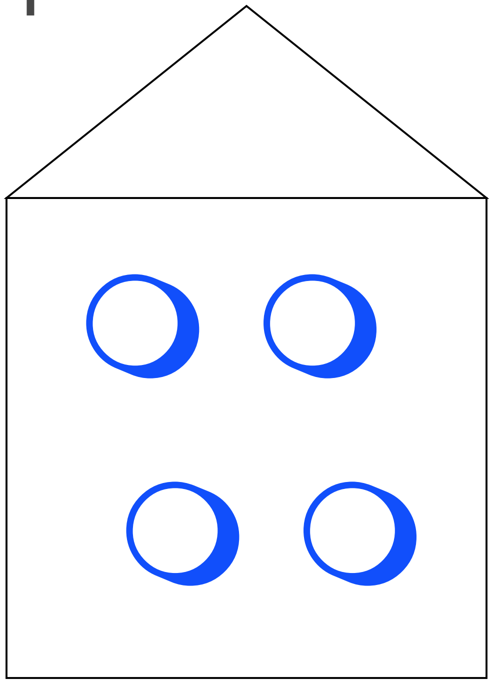
$(g \circ f)(n) = g(f(n)) = g(n+1) = (n+1)^2$

$(f \circ g)(n) = f(g(n)) = f(n^2) = n^2 + 1$

Note: $g \circ f \neq f \circ g$

The pigeonhole principle





- Basic form:

  A function from one finite set to a smaller finite set cannot be one-to-one; there must be at least two elements in the domain that have the same image in the codomain.

Pigeon Hole Principle

- Using this class as the domain:
  - must two people share a birth month?
  - must two people share a birthday?
- There exist two people in New York City who are not bald, and who have the same number of hairs on their heads.
- Pairs of people are either acquainted with one another or not. In any group of people, there must be two people who have the same number of acquaintances within the group. TRICKY!

Generalized Pigeon Hole Principle

13 donuts

3 Homers

Claim: One of the Homers will get at least 5 donuts.

Why? [# of Donuts] > 4 * [# of Homers]

Generalized Pigeon Hole Principle

The generalized pigeonhole principle:

For any function f from a finite set X to a finite set Y: if $n(X) > k * n(Y)$, then there is some $y \in Y$ such that y is the image of at least k+1 distinct elements of X.

Examples:

- Estimate the number of people in the room. What is the minimum number of people who all share the same birth month?
- There are 35 baskets of apples. Each basket has 20 to 30 apples. Show that there are at least 4 baskets containing the same number of apples.