MATH299M/CMSC389W
Spring 2019 – Ajeet Gary, Devan Tamot, Vlad Dobrin
Model H2: Convergence of Fibonacci Sequence Ratio
Assigned: Wednesday February 6th
Due: Monday February 25th, 11:59PM

Welcome to your second homework assignment! Like the last, I want to keep it as open-ended as possible. I'm going to throw a bunch of math at you, sprinkled with ideas of stuff you can implement, you don't have to do all of it, just make something cool that teaches you something! Also: Week 6 when we learn how to use Graphics we're going to revisit Fibonacci and make some pictures, it's going to be a lot of fun, and paying attention to this project will get you excited for that one :D

You've probably heard of the *Fibonacci Sequence*. It's a recursive sequence that starts with two 1's, and then each successive term is the sum of the last two:

$$F(n) = \begin{cases} 1, & n = 1 \\ 1, & n = 2 \\ F(n-1) + F(n-2), & n > 2 \end{cases}$$

The Fibonacci Sequence is awesome! It has so many cool properties. It's also fun to write the code for it as a simple recursive function – yes, Mathematica already has a function called Fibonacci, but it's still fun to make your own. One is that the ratio of the terms converges to a certain number called *The Golden Ratio*, denoted with the Greek letter φ. The Golden Ratio is equal to $(\sqrt{5}+1)/2 = 1.6180$ ….

$$\lim_{n\to\infty} \frac{F(n+1)}{F(n)} = \frac{\sqrt{5}+1}{2}$$

This would be cool to see, maybe also a visualization of how fast it approaches that ratio. This ratio is the most aesthetic ratio for the side lengths of a rectangle to our eyes, you can test for yourself: a MacBook Pro's screen dimensions are 2560 x 1600 = 1.6. Amazingly the Parthenon is a Golden Rectangle (a rectangle for which the length/width = φ), and it was built long before the time of Fibonacci, evidence of this ratio's natural aesthetic significance!

We can generate the Golden Ratio a different way, through something called *continued fractions*. A continued fraction is an infinite fraction, they look like this:

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cfrac{1}{\ddots}}}}}$$

The Golden Ratio is equal to the continued fraction where all of the a_i are equal to 1. Can you think of a way to approximate this value? Hint: Use a recursive function that nests your fraction deeper and deeper. Hint's hint: Use the Nest or NestList functions; Nest[f,a,b] returns $f^b(a)$, that is, iterating the function, and NestList returns a list of values {a,f(a),f(f(a)),…}, that is, the iteration at each step. The Golden Ratio is the hardest number to approximate with a fraction, which is why the a_i are always 1; as you nest the fraction deeper and deeper, it creeps towards φ slower than any other continued fraction. For this reason you see the Fibonacci Sequence in petals of flowers, that is, each level will have a Fibonacci Sequence number of petals – to maximize the light that hits the petals, you want to minimize overlap, which means it's in your best interest if the ratio between number of petals level to level isn't a simple rational number, because then you'd get overlap, like if one level has 18 petals and the next down had 24, for example, then you'd get 6 petals on the bottom layer that are totally covered!

There's a similar sequence called the *Lucas Numbers*, which is the same but starts with 1,2 instead of 1,1. You could adapt your code for the Fibonacci numbers to make one for the Lucas numbers, and compare how they grow. It should be clear that you can actually easily adapt the code to model any recursively defined sequence, like the Pell numbers:

$$Pell(n) = \begin{cases} 0, & n = 1 \\ 1, & n = 2 \\ 2 * Pell(n-1) + Pell(n-2), & n > 2 \end{cases}$$

The ratio of these converges to the so-called *Silver Ratio*, equal to $1 + \sqrt{2}$, this would also be cool to see. Do the Lucas numbers converge to anything?

At this point you've made tools for recursive sequences and continued fractions, you should play around with them! Euler's number is also a cool continued fraction; the notation for the continued fraction representation of a number is to put the list of numbers in square brackets delimited by commas, except for between the first two numbers, where you use a semi-colon:

$$e = [2; 1,2,1,1,4,1,1,6,1,1,8, \dots]$$

Here's pi:

$$\pi = [3; 7,15,1,292,1,1,1,2,1,3,1, \dots]$$

Note: you can look up sequences like this in *The On-Line Encyclopedia of Integer Sequences*; e's continued fraction is sequence A003417 - https://oeis.org/A003417.