MATH299M/CMSC389W – *Visualization Through Mathematica*
Spring 2019 – Ajeet Gary, Devan Tamot, Vlad Dobrin
Model H8.1: Visualizing Trig Functions on the Complex Plane
Assigned: Friday March 15th, 2019
Due: Monday April 1st, 2019 11:59PM

Note: Between models H7.1, H7.2, H8.1, H8.2, H9.1, and H9.2 (Group 2) you need only complete 3 assignments.

You're familiar with the six standard trigonometric functions sine, cosine, tangent, cotangent, secant, and cosecant from grade school. You're also familiar with the inverse functions for each of these, arcsine, arccosine, etc. What you probably haven't seen is that these functions can take *complex* values. Try this for yourself! If you input Sin[z] into Mathematica where z is some complex number, it won't throw an error, it will give you a complex number back. The trigonometric functions are really functions from the complex plane to the complex plane, and it just so happens that on the domain of the real line the outputs are all real as well, which is what allows us to restrict our attention to the real slice in grade school.

Now, plotting complex functions isn't straightforward, since assigned to each point in the complex plane is a *complex* number, not just a number on the number line. This means that you can't make a surface like you can for the real numbers. In principal nothing is different, in fact some mathematicians refer to the complex plane as the *complex line* because it's really just 1 dimension of complex numbers, however visually we aren't able to accommodate this living in 3-space.

A technique for visualizing complex functions is to draw curves on the complex plane and then on a side-by-side plot show the image of those curves transformed under the function. Some illuminating choices for these curves are vertical lines, horizontal lines, radial lines, and concentric circles. Combining the first two of these gives you a Euclidian grid, combining the second two gives you the grid of a polar graph. You can also plot individual points and see where they go. I suggest coloring these curves so that you can tell which images correspond to which preimages.

Once you make these visualizations, I highly recommend zooming out, you'll find some extremely surprising behavior for the trig functions!

If you haven't dealt with complex functions much (or at all) before, this is a good opportunity to investigate how the complex plane works. I suggest plotting $re^{i\theta}$ with sliders for r and θ and see how the point moves around the plane.

Note that Mathematica doesn't have anything that might reasonably be called "ComplexPlot", you're going to have to break up a complex number into its real and imaginary parts using the functions Re and Im, and then glue them back together for calculations.

Other things to investigate:

Euler's Formula

Euler's formula is an extremely interesting relation between complex numbers, Euler's constant, and trigonometric functions:

$$e^{i\theta} = cos\theta + i\sin\theta$$

This is extremely mysterious! Can your models shed some light on it? Try plotting the exponential function, and try plotting complex linear combinations of trig functions.

Hyperbolic Trigonometric Functions
There are some more trig functions that you don't learn about in grade school, the *hyperbolic trigonometric functions*. They are sinh (pronounced "sinch"), cosh (pronounced "cahsh"), tanh, coth, sech, and csch. The hyperbolic trig function are related to the regular trig functions through the following relations:

$$sinh(\theta) = -isin(i\theta)$$
$$cosh(\theta) = cos(i\theta)$$
$$tanh(\theta) = -itan(i\theta)$$
$$coth(\theta) = icot(i\theta)$$
$$sech(\theta) = sec(i\theta)$$
$$csch(\theta) = icsc(i\theta)$$

Can you figure out what these look like, and perhaps what they're good for?

Derivatives of Trig Functions
Why is the derivative of Sine, Cosine? Is this illuminated more on the complex plane? Can you see a geometric reason why the fourth derivative of Sine should be itself? What about the derivative of Tangent being Secant^2? Derivative of Cotangent is negative Cosecant^2?

Trigonometric Identities
In grade school you learned many trig identities, namely the *Pythagorean identities*, *angle sum and difference identities* (same thing), *double angle identities* (special case of addition identities), *half angle identities* (derived from double angle identities), *parity identities* (just based on evenness/oddness of the functions), and the *phase shift identities*. You may have proved these using each other – a cool thing to play around with is converting these to exponential form and then showing the identities that way (it's much easier). Here's what Sine and Cosine look like in exponential form:

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

You can easily check these using Euler's formula. Can you use the complex definitions of the trig functions to gain some insight into why these identities work?