MATH299M/CMSC389W – *Visualization Through Mathematica*
Spring 2019 – Ajeet Gary, Devan Tamot, Vlad Dobrin
Project 2a: Partial Derivatives and Gradient Visualization
Assigned: Friday March 15^{th}
Due: Monday April 15^{th}, 11:59PM

Now that you have plenty of visualization tools, in both two dimensions and three dimensions, under your belt for graphing functions, you're equipped to start doing some awesome math stuff. For the second project you need only complete one of Project 2a (this one), Project 2b, or Project 2c. If you like multivariate calculus, this is one is a good choice.
Note: if you feel unprepared mathematically for any of these assignments, let us know and we will assign you an assignment based in Calculus II material.

Like all assignments in this class, I'm going to leave the project fairly open-ended. I'll go through a review of partial derivatives and gradients (basic multivariate calculus tools) and then list some ideas of where your project could go.

In Calculus I you take derivatives of functions of a single variable:

$$\frac{d}{dx}f(x) = f'(x)$$

the geometric interpretation being that f'(x) is the slope of the tangent line to f(x) at the point x. Intuitively this is the rate of change of the independent variable wrt the dependent variable – if you're a physics major (or even not) your go-to example is surely x(t), the position function in terms of time, making the first derivative x'(t) interpretable as the velocity, that is, the rate of change of position wrt time.

In Multivariate Calculus we deal with functions of multiple variables. These are functions that depend on multiple dependent variables, assigning a real number to each pair or triple of numbers for functions of two or three variables, respectively (of course we have n variables, but Multivariate Calculus courses generally only play with the dimensions we can see). When you take the derivative of a function of multiple variables, you can do it wrt any of the variables it's defined in terms of. It makes sense that we want to be able to do this because we want to analyze how our function of multiple variables changes wrt each variable. To take the *partial derivative* of a function wrt one of its variables you simply hold the other variables constant:

$$\frac{\partial}{\partial x}f(x,y) = f_x(x,y)$$

the geometric interpretation being that this is the slope of the tangent *plane* to the surface f(x,y) in the direction of x. Curiously, for a continuous functions the second partial derivatives of a function where the derivatives are wrt different variables, the so called *mixed partials*, are equal:

$$f(x,y)\ continuous\ \Longrightarrow \frac{\partial}{\partial x}\frac{\partial}{\partial y}f(x,y) = f_{yx}(x,y) = f_{xy}(x,y) = \frac{\partial}{\partial y}\frac{\partial}{\partial x}f(x,y)$$

Why is this? In fact, what even is the geometric interpretation of a mixed partial derivative? Make a visualization that shows why it should be obvious that continuous functions have this property. You may want to start by showing what a single derivative and second derivative wrt the same variable look like for a function of two variables to begin with, that is, on a surface.

Partial derivatives only describe how a function of two variables in changing in one direction. If we want a better image of how the function is changing over its domain, we can collect all of the first partials in a vector called the *gradient*:

$$\nabla f(x,y) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\cdot f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

Looking at this vector tells you how fast the function is increasing or decreasing in each direction, and more generally lets you deduce the rate of change in an arbitrary direction by computing the *directional derivative*:

$$D_{\hat{v}}f(x,y) = \nabla f(x,y)\cdot \hat{v}$$

which you get by taking the dot product of the gradient with the unit vector $\hat{v}$ that denotes the direction for which you would like to know how the function is changing. If you paid attention in your Multivariate Calculus class and this makes sense to you already – great! Make a visualization of it! If it doesn't make sense to you yet – also great! Visualize it and get a grasp of it! I think a directional derivatives model that lets you specify the direction and shows you what that looks like would be very cool.

The gradient has an interesting property – it points in the direction of fastest descent on the surface. Can we see this?

Other cool Multivariate Calculus ideas with derivatives of functions of multiple variables:

Vector fields

The gradient of a function at a point gives you a vector at that point that points in the direction the function is changing fastest. If you take the gradient of a function of multiple variables in general, you get a new function that takes in those variables and gives back out vectors – we call this a *vector field*. Can we visualize this for two and three dimensions? How about the (non-mixed) second derivatives?

Divergence

The *divergence* of a function is defined as:

$$\nabla \cdot f(x,y,z) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot f(x,y,z) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$$

which can be thought of as the dot product of the gradient *operator* (notice that the 3-tuple on the left side of the · contains operators, that is, things that do things to functions, rather than functions or variables themselves). What we get out is a *scalar field* – can you make a model that shows the divergence of a function of two or three variables, and illuminates what this real number assigned to each point really mean?

Curl

The *curl* of a function is defined as:

$$\nabla \times f(x,y,z) = \left(f_{zy}, f_{xz}, f_{yx}\right)$$

which can be interpreted a cross product of the partial derivative operators and the first partials of the function. You'll get a vector field, a vector assigned to each point two-space or three-space. Can you make a model that shows this, and illuminates the physical meaning of the curl?

Hint:

My go-to continuous function of three variables is a temperature function, that is, a function T(x,y,z) that gives the temperature of the space at the point (x,y,z). This might be high near a heat vent and then continuously decrease as you move across the room, in a linear way if the air is still or with a non-zero curl if someone turned a fan on to induce a current in the room. This is a good intuitive tool for two dimensions as well. DensityPlot even has a ColorFunction preset called "Temperature".

Good luck! Make something awesome that they could (nay, should) use in a Multivariate Calculus class. The final product should be cleaned up, that is, there should be a title, your name and the date on it, text in Text cells, and no extraneous code or outputs floating around. After making your model, you should play around with it, and maybe in a Text cell describe some interesting behavior you found.